Enumeration of I-graphs: Burnside does it again

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Abstract

We give explicit and efficiently computable formulas for the number of isomorphism classes of I-graphs, connected I-graphs, bipartite connected I-graphs, generalized Petersen graphs, and bipartite generalized Petersen graphs. The tool that we use is the well-known Cauchy-Frobenius-Burnside lemma.

Keywords: I-graphs, generalized Petersen graphs, Cauchy-Frobenius-Burnside lemma, arithmetical functions.

Math. Subj. Class.: 05A15, 05C30

1 Introduction

Recently the class of I-graphs, introduced in the Foster Census [2] as a further development of the generalized Petersen graphs, has received considerable attention. One reason for this is that bipartite I-graphs give rise to some highly symmetric configurations of points and lines [1]. In the same paper, Boben, Pisanski and Žitnik characterized the automorphism groups of those I-graphs which are not generalized Petersen graphs, so that together with the earlier results of Frucht, Graver and Watkins [3], the characterization of the automorphism groups of I-graphs is now complete. Finally, Horvat, Pisanski and Žitnik have recently shown that every I-graph has a nondegenerate unit-distance representation in the Euclidean plane [4]. This answers the question of whether every generalized Petersen...
graph can be drawn in the plane in such a way that all edges are represented by straight-line segments of equal length.

As witnessed by the recent inclusion of the corresponding counting sequences in [10], there has also been interest in the enumeration of non-isomorphic I-graphs and various of their subclasses, such as connected I-graphs, generalized Petersen graphs, etc. However, explicit formulas for the $n$-th term of these sequences seem to be unknown, with the sole exception of the formula for the number of non-isomorphic generalized Petersen graphs $G(n, k)$ on $2n$ vertices with $\gcd(n, k) = 1$, given quite recently by Steimle and Staton [12, Thm. 11].

At a seminar meeting in Ljubljana in January 2009, T. Pisanski asked for a formula enumerating non-isomorphic I-graphs on $2n$ vertices. We give such a formula below in Section 2, as well as analogous formulas enumerating non-isomorphic connected I-graphs, bipartite connected I-graphs, generalized Petersen graphs, and bipartite generalized Petersen graphs on $2n$ vertices. These formulas are in closed form, and can be used for efficient computation of the number of isomorphism classes, provided that the prime factorization of $n$ is known.

To enumerate isomorphism classes we use the Cauchy-Frobenius lemma, also known as Burnside’s lemma. Although very well known, this lemma is seldom applied directly, but rather indirectly via the Redfield-Pólya enumeration theorem whose proof relies on it. Recently, though, it has been used successfully on its own in several cases (cf. [9, 6, 7]).

For $n \in \mathbb{N}$ write $\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}$ and $\mathbb{Z}_n' = \mathbb{Z}_n \setminus \{0, n/2\}$. Let $n \in \mathbb{N}$, $n \geq 3$, and $j, k \in \mathbb{Z}_n'$. The I-graph $I(n, j, k)$ is the graph $G = (V, E)$ where

$$V = \mathbb{Z}_n \times \mathbb{Z}_2,$$

$$E = \bigcup_{i=0}^{n-1} \{(i, 0), (i, 1)\}, \{(i, 0), (i + j, 0)\}, \{(i, 1), (i + k, 1)\},$$

and addition is performed modulo $n$. Well-known special cases include the $n$-prism $Y_n = I(n, 1, 1)$, the Petersen graph $I(5, 1, 2)$, and the generalized Petersen graph $G(n, k) = I(n, 1, k)$, introduced by Watkins in [13].

The I-graph $I(n, j, k)$ is a cubic graph on $2n$ vertices. In [1], several graph-theoretic properties of $I(n, j, k)$ such as connectedness, girth, being bipartite or being vertex-symmetric, are characterized in terms of number-theoretic properties of parameters $n, j, k$. An algorithm for deciding which sets of parameter values give rise to isomorphic I-graphs is also given there. In [5], the following result (crucial for our enumeration) is proved:

**Theorem 1.1.** $I(n, j, k)$ and $I(n, j', k')$ are isomorphic if and only if there exists an integer $a$, relatively prime to $n$, such that either \{\{j', k'\} = \{aj \mod n, ak \mod n\}$ or \{\{j', k'\} = \{aj \mod n, -ak \mod n\}.

We also rely on the following results from [1]:

**Theorem 1.2.** The graph $I(n, j, k)$ is connected if and only if $\gcd(n, j, k) = 1$.

**Theorem 1.3.** A connected graph $I(n, j, k)$ is bipartite if and only if $n$ is even and $j$ and $k$ are odd.

In the rest of the paper, we use the following notation (for $n \in \mathbb{N}$):
\[ I(n) = \text{the number of isomorphism classes of I-graphs } I(n, j, k) \]
\[ I_c(n) = \text{the number of isomorphism classes of connected I-graphs } I(n, j, k) \]
\[ I_{bc}(n) = \text{the number of isomorphism classes of bipartite connected I-graphs } I(n, j, k) \]
\[ P(n) = \text{the number of isomorphism classes of generalized Petersen graphs } G(n, k) = I(n, 1, k) \]
\[ P_b(n) = \text{the number of isomorphism classes of bipartite generalized Petersen graphs } G(n, k) = I(n, 1, k) \]
\[ P_{r}(n) = \text{the number of isomorphism classes of generalized Petersen graphs } G(n, k) = I(n, 1, k) \text{ with } \gcd(n, k) = 1 \]
\[ \mathbb{Z}_n = \{0, 1, \ldots, n - 1\} \quad \text{(the ring of integers modulo } n) \]
\[ \mathbb{Z}_n^* = \{a \in \mathbb{Z}_n; \gcd(a, n) = 1\} \quad \text{(the group of units of } \mathbb{Z}_n) \]
\[ \mathbb{Z}_n' = \mathbb{Z}_n \setminus \{0, n/2\} \quad \text{(the set of legal values for } j, k \text{ in } I(n, j, k)\) \]

For \( k \in \mathbb{Z} \), we write \( k \mod n \) to denote the unique \( r \in \mathbb{Z}_n \) such that \( k \equiv r \pmod{n} \).

In particular, if \( n \) is even, then
\[
(n/2) \mod 2 = \begin{cases} 0, & n \equiv 0 \pmod{4}, \\ 1, & n \equiv 2 \pmod{4}. \end{cases}
\]

Table 1 lists the arithmetical functions that appear in the rest of the paper. The column “OEIS id” in Table 1 gives the corresponding identifier from [10].

<table>
<thead>
<tr>
<th>notation</th>
<th>OEIS id</th>
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<td>( \mu(n) )</td>
<td>A008683</td>
<td>Moebius function</td>
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<td>( \tau(n) )</td>
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<td>the number of divisors of ( n )</td>
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<td>( \varphi(n) )</td>
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<td>Euler’s totient function, ( \varphi(n) =</td>
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<td>( J_2(n) )</td>
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<td>the second Jordan’s totient function, ( J_2(n) =</td>
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<td>the number of distinct prime factors of ( n )</td>
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<td>A060594</td>
<td>the number of square roots of ( 1 ) modulo ( n ), ( r(n) =</td>
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<tr>
<td>( s(n) )</td>
<td>A000089</td>
<td>the number of square roots of ( -1 ) modulo ( n ), ( s(n) =</td>
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Table 1: Some arithmetical functions.

With the exception of \( \omega(n) \) which is additive, all other functions in Table 1 are multiplicative. If \( p \) is a prime and \( k \geq 1 \), we have
\[
J_2(p^k) = p^{2k} - p^{2k-2} = \sum_{d | p^k} \mu \left( \frac{p^k}{d} \right) d^2,
\]
\[ r(p^k) = \begin{cases} 1, & p = 2 \text{ and } k = 1, \\ 2, & p \text{ odd or } (p = 2 \text{ and } k = 2), \\ 4, & p = 2 \text{ and } k \geq 3, \end{cases} \]

\[ s(p^k) = \begin{cases} 0, & p \equiv 3 \pmod{4} \text{ or } (p = 2 \text{ and } k \geq 2), \\ 1, & p = 2 \text{ and } k = 1, \\ 2, & p \equiv 1 \pmod{4}, \end{cases} \]

hence

\[ J_2(n) = n^2 \prod_{p \mid n, p \text{ prime}} \left(1 - \frac{1}{p^2}\right) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) d^2, \]

\[ r(n) = \begin{cases} 2^{\omega(n)/2}, & n \equiv 1 \pmod{2} \text{ or } n \equiv 4 \pmod{8}, \\ 2^{\omega(n)-1}, & n \equiv 2 \pmod{4}, \\ 2^{\omega(n)+1}, & n \equiv 0 \pmod{8}, \end{cases} \]

\[ s(n) = \begin{cases} 0, & 4 \mid n \text{ or } \exists p \text{ prime} : (p \mid n \text{ and } p \equiv 3 \pmod{4}), \\ 2^{\psi(n)}, & \text{otherwise}, \end{cases} \]

where \( \psi(n) = |\{p \mid n; p \text{ prime}, p \equiv 1 \pmod{4}\}|. \)

The following formula (which can also be proved by our methods) is given in \cite[Thm. 11]{12}:

**Theorem 1.4.** The number \( P_r(n) \) of isomorphism classes of generalized Petersen graphs \( G(n,k) \) on \( 2n \) vertices with \( \gcd(n,k) = 1 \) is given by

\[ P_r(n) = \frac{1}{4}(\varphi(n) + r(n) + s(n)). \quad (1.1) \]

In Section 2 we list our formulas for \( I(n), I_c(n), I_{bc}(n), P(n), P_b(n) \) which seem to be new, and tabulate their values (as well as those of \( P_r(n) \)) for some small values of \( n \). In Section 3 we explain our proof techniques and give the proofs.

## 2 The main results

**Theorem 2.1.** Let \( n = p_1^{k_1}p_2^{k_2} \cdots p_{\omega(n)}^{k_{\omega(n)}} \) be the prime factorization of \( n \). Then the number of isomorphism classes of I-graphs on \( 2n \) vertices is given by

\[ I(n) = \frac{1}{4} \sum_{i=1}^{\omega(n)} \prod_{j=1}^{\omega(n)} g_i\left(p_j^{k_j}\right) - \begin{cases} 2\tau(n) - 1, & n \text{ even}, \\ \tau(n), & n \text{ odd}, \end{cases} \quad (2.1) \]
where

\[
g_1(p^k) = \frac{(p+1)p^k - 2}{p-1}, \quad (2.2)
\]

\[
g_2(p^k) = \begin{cases} 
4k, & p = 2, \\
2k + 1, & p > 2,
\end{cases} \quad (2.3)
\]

\[
g_3(p^k) = \begin{cases} 
2, & p = 2 \text{ and } k = 1, \\
4(k-1), & p = 2 \text{ and } k \geq 2, \\
2k + 1, & p > 2,
\end{cases} \quad (2.4)
\]

\[
g_4(p^k) = \begin{cases} 
2, & p = 2, \\
2k + 1, & p \equiv 1 \pmod{4}, \\
1, & p \equiv 3 \pmod{4}.
\end{cases} \quad (2.5)
\]

**Theorem 2.2.** The number \( P(n) \) of isomorphism classes of generalized Petersen graphs on \( 2n \) vertices is given by

\[
P(n) = \frac{1}{4}(2n - \varphi(n) - 2 \gcd(n, 2) + r(n) + s(n)). \quad (2.6)
\]

**Theorem 2.3.** The number of isomorphism classes of connected I-graphs on \( 2n \) vertices is given by

\[
I_c(n) = \frac{1}{4} \left( \frac{J_2(n)}{\varphi(n)} + r(n) + s(n) + t(n) \right) - \begin{cases} 
1, & n \text{ odd}, \\
2, & n \equiv 0 \pmod{4}, \\
3, & n \equiv 2 \pmod{4}
\end{cases} \quad (2.7)
\]

where

\[
t(n) = \begin{cases} 
2^{\omega(n)} + 2^{\omega(n/2)}, & n \text{ even}, \\
2^{\omega(n)}, & n \text{ odd}.
\end{cases} \quad (2.8)
\]

**Theorem 2.4.** For \( n \) even, let \( \chi(n) = (n/2) \pmod{2} \). The number of isomorphism classes of bipartite generalized Petersen graphs on \( 2n \) vertices is given by

\[
P_b(n) = \begin{cases} 
\frac{1}{4}(n - \varphi(n) - 2\chi(n) + r(n) + s(n)), & n \text{ even} \\
0, & n \text{ odd}
\end{cases} \quad (2.9)
\]

**Theorem 2.5.** For \( n \) even, let \( \chi(n) = (n/2) \pmod{2} \). The number of isomorphism classes of bipartite connected I-graphs on \( 2n \) vertices is given by

\[
I_{bc}(n) = \begin{cases} 
\frac{1}{4} \left( \frac{J_2(n)}{3\varphi(n)} + \chi(n)2^{\omega(n/2)} + r(n) + s(n) \right) - \chi(n), & n \text{ even} \\
0, & n \text{ odd}
\end{cases} \quad (2.10)
\]

**Corollary 2.6.** Let \( p \) be an odd prime. Then

\[
I(p) = I_c(p) = P(p) = P_r(p) = \left\lfloor \frac{p}{4} \right\rfloor.
\]
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Table 2: The values of $I(n)$, $I_c(n)$, $P(n)$, $P_r(n)$ for $3 \leq n \leq 103$. 
Table 3: The values of $I(n)$, $I_c(n)$, $P(n)$, $P_r(n)$ for $105 \leq n \leq 142$.

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Table 4: The values of $I_{bc}(2n)$ and $P_b(2n)$ for $2 \leq n \leq 76$.

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| $n$   | 38  | 40  | 42  | 44  | 46  | 48  | 50  | 52  | 54  | 56  | 58  | 60  | 62  | 64  | 66  | 68  |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $I_{bc}(n)$ | 5   | 8   | 9   | 7   | 6   | 10  | 8   | 8   | 9   | 10  | 8   | 14  | 8   | 9   | 13  | 10  |
| $P_b(n)$ | 5   | 8   | 8   | 7   | 6   | 10  | 8   | 8   | 9   | 10  | 8   | 13  | 8   | 9   | 12  | 10  |

| $n$   | 70  | 72  | 74  | 76  | 78  | 80  | 82  | 84  | 86  | 88  | 90  | 92  | 94  | 96  | 98  | 100 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $I_{bc}(n)$ | 13  | 14  | 10  | 11  | 15  | 14  | 11  | 18  | 11  | 14  | 19  | 13  | 12  | 18  | 14  | 16  |
| $P_b(n)$ | 12  | 14  | 10  | 11  | 14  | 14  | 11  | 17  | 11  | 14  | 17  | 13  | 12  | 18  | 14  | 16  |

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3 The proofs

3.1 The Burnside technology

Let $\alpha$ be the action of a finite group $G$ on a finite set $A$. Then we denote by $\sim_{\alpha}$ the associated equivalence relation on $A$, by $|A/\sim_{\alpha}|$ the number of orbits of $\alpha$, and by $\text{fix}_{\alpha}(g)$ the number of elements of $A$ fixed by $g \in G$ under $\alpha$. Our main enumeration tool is the Cauchy-Frobenius-Burnside lemma:

Lemma 3.1.

$$|A/\sim_{\alpha}| = \frac{1}{|G|} \sum_{g \in G} \text{fix}_{\alpha}(g).$$

For a proof, see, e.g., [11, Lemma 7.24.5]).

First we list some auxiliary results which will be useful in the sequel.

Proposition 3.2. Let $\vartheta_n$ be the multiplicative action of $\mathbb{Z}_n^*$ on $\mathbb{Z}_n$. Then

$$|\mathbb{Z}_n/\sim_{\vartheta_n}| = \frac{1}{\varphi(n)} \sum_{a \in \mathbb{Z}_n^*} \gcd(n, a - 1).$$  (3.1)

Proof. Assume that $j \in \mathbb{Z}_n$, $a \in \mathbb{Z}_n^*$, $d = \gcd(n, a - 1)$, $n = n'd$ and $a - 1 = a'd$. Then $\gcd(n', a') = 1$, and so $j$ is fixed by $a$ iff

$$aj \equiv j \pmod{n} \iff n | (a - 1)j \iff n' | a'j \iff n' | j.$$

It follows that the set of $j$ fixed by $a$ is $\{0, n', 2n', \ldots, (d - 1)n'\}$, hence $\text{fix}_{\vartheta}(a) = d = \gcd(n, a - 1)$, and Lemma 3.1 gives (3.1). $\square$

Lemma 3.3. Let $a, d, n \in \mathbb{N}$ be such that $d | n$ and $\gcd(a, d) = 1$. Then there is an $x \in \mathbb{Z}$ such that $\gcd(a + xd, n) = 1$.

Proof. Let $x \in \mathbb{Z}_n$ satisfy

$$x \not\equiv -a d^{-1} \pmod{p}$$

for each prime $p$ which divides $n$ but not $d$. Note that $d$ is invertible mod $p$ for such $p$, and that such an $x$ exists by the Chinese Remainder Theorem.

Assume that $\gcd(a + xd, n) \neq 1$. Then there exists a prime $p$ such that $p | n$ and $p | (a + xd)$. We distinguish two cases.

a) If $p | d$ then $p | a$, contrary to the assumption that $\gcd(a, d) = 1$.

b) If $p \nmid d$ then

$$a + xd \equiv 0 \pmod{p} \implies x \equiv -a d^{-1} \pmod{p},$$

contrary to the choice of $x$.

In either case we reach a contradiction, hence $\gcd(a + xd, n) = 1$. $\square$

Corollary 3.4. Let $\vartheta_n$ be as in Proposition 3.2. For all $j, k \in \mathbb{Z}_n$ we have:

(i) $j \sim_{\vartheta_n} \gcd(n, j)$,
(ii) \( j \sim_{\theta} k \iff \gcd(n, j) = \gcd(n, k) \),

(iii) each orbit of \( \theta_n \) contains exactly one positive divisor of \( n \) (with \( n \) replaced by 0), and \( |\mathbb{Z}_n/\sim_{\theta_n}| = \tau(n) \).

**Proof.** (i) Let \( d = \gcd(n, j), n' = n/d, j' = j/d \). Then \( \gcd(n', j') = 1 \), so there are \( a', k \in \mathbb{Z} \) such that \( a'j' = 1 + kn' \). Since \( \gcd(a', n') = 1 \) and \( n' \mid n \), Lemma 3.3 implies that there is an \( x \in \mathbb{Z} \) such that \( a := a' + xn' \in \mathbb{Z}_n^* \). Then

\[
aj = (a' + xn')j'd = a'j'd + xj'n = (1 + kn')d + xj'n = d + (k + xj')n,
\]

hence \( aj \equiv d \pmod{n} \). So \( j \sim_{\theta} d \), proving the claim.

(ii) Let \( j \sim_{\theta} k \). Then there are \( a \in \mathbb{Z}_n^* \) and \( m \in \mathbb{Z} \) such that \( aj - k = mn \). This implies that any common divisor of \( j \) and \( n \) divides \( k \), and any common divisor of \( k \) and \( n \) divides \( aj \) and hence \( j \). It follows that \( \gcd(n, j) = \gcd(n, k) \).

Conversely, let \( \gcd(n, j) = \gcd(n, k) \). Then by (i), \( j \sim_{\theta} k \).

(iii) By (i), each orbit of \( \sim_{\theta} \) contains a positive divisor of \( n \) (with \( n \) replaced by 0). By (ii), different positive divisors of \( n \) (with \( n \) replaced by 0) belong to different orbits of \( \sim_{\theta} \). This proves the claim. \( \square \)

**Lemma 3.5.** Let \( a, b, c \in \mathbb{Z}, n, k \in \mathbb{N} \).

(i) If \( a \equiv b \pmod{n} \) then \( \gcd(a, n) = \gcd(b, n) \).

(ii) If \( \gcd(a, b) = 1 \) then \( \gcd(ab, c) = \gcd(a, c) \gcd(b, c) \).

(iii) Any set of \( nk \) consecutive integers contains exactly \( k \) multiples of \( n \).

The straightforward proofs are omitted.

Now we embark on our main task of enumerating isomorphism classes of I-graphs. For a fixed \( n \geq 3 \), we represent the I-graph \( I(n, j, k) \) with the ordered pair \((j, k)\). We need to construct a suitable group \( G_n \) acting on the set \( \mathbb{Z}_n \times \mathbb{Z}_n \) in such a way that the orbits of this action will be in one-to-one correspondence with the isomorphism classes of I-graphs. In view of Theorem 1.1, the following choice is natural.

**Definition 3.6.** By \( G_n \) we denote the subgroup of the symmetric group \( S(\mathbb{Z}_n \times \mathbb{Z}_n) \) generated by the permutations \((\xi_a)_{a \in \mathbb{Z}_n^*}, \mu, \rho: \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n \times \mathbb{Z}_n \), where for all \( a \in \mathbb{Z}_n^* \) and \((j, k) \in \mathbb{Z}_n \times \mathbb{Z}_n \):

\[
\xi_a(j, k) \equiv (aj, ak) \pmod{n}, \\
\mu(j, k) \equiv (j, -k) \pmod{n}, \\
\rho(j, k) \equiv (k, j) \pmod{n}.
\]

**Proposition 3.7.**

\[
G_n = \{ \xi_a, \xi_a\mu, \xi_a\mu, \xi_a\rho; a \in \mathbb{Z}_n^* \} \tag{3.2}
\]

and \( |G_n| = 4\varphi(n) \).

**Proof.** It is straightforward to check that for all \( a, b \in \mathbb{Z}_n^* \),

\[
\xi_a\xi_b = \xi_{ab}, \\
\xi_a\xi_a^{-1} = \xi_1 = \text{id}_{\mathbb{Z}_n \times \mathbb{Z}_n} = \mu^2 = \mu^2, \\
\mu\xi_a = \xi_a\mu, \\
\rho\xi_a = \xi_a\rho, \\
\mu\rho = \xi_{-1}\rho\mu.
\]
Using these equalities we can show that for any $g \in G_n$ there are $a \in \mathbb{Z}_n^*$ and $\epsilon, \delta \in \{0, 1\}$ such that

$$g = \xi_a \rho^\epsilon \mu^\delta,$$

which proves (3.2). Now write $g_i = \xi_a, \rho^\epsilon \mu^\delta_i$ for $i \in \{1, 2\}$. Assume that $g_1 = g_2$, and compute

$$g_i(1, 1) = \begin{cases} (a_i, (-1)^\delta a_i), & \epsilon_i = 0, \\ ((-1)^\delta a_i, a_i), & \epsilon_i = 1. \end{cases}$$

If $\epsilon_1 \neq \epsilon_2$, then $g_1(1, 1) = g_2(1, 1)$ implies that $a_1 = (-1)^\delta_2 a_2$ and $a_2 = (-1)^\delta_1 a_1$, hence $a_1 = (-1)^{\delta_1+\delta_2} a_1$. Cancelling $a_1$ yields $(-1)^{\delta_1+\delta_2} = 1$, and so $\delta_1 = \delta_2$. W.l.g. assume that $\epsilon_1 = 1$ and $\epsilon_2 = 0$. Then $g_1 = g_2$ turns into $\xi_{\alpha_1} \rho = \xi_{\alpha_2}$. Applying both sides of this equality to $(1, 1)$ yields $(a_1, a_1) = (a_2, a_2)$, hence $a_1 = a_2$ and $\xi_{\alpha_1} = \xi_{\alpha_2}$. Now $\xi_{\alpha_1} \rho = \xi_{\alpha_2}$ implies $\rho = \xi_1$. On the other hand, the initial assumption that $n \geq 3$ implies that $|\mathbb{Z}_n^*| \geq 2$, hence $\rho \neq \xi_1$.

This contradiction shows that $\epsilon_1 = \epsilon_2$. Then $g_1(1, 1) = g_2(1, 1)$ implies that $a_1 = a_2$ and $(-1)^\delta a_1 = (-1)^\delta a_2$, hence $(-1)^{\delta_1} = (-1)^{\delta_2}$, and so $\delta_1 = \delta_2$.

We have shown that $g_1 = g_2$ if and only if $a_1 = a_2$ and $\epsilon_1 = \epsilon_2$ and $\delta_1 = \delta_2$. Hence $|G_n| = 4|\mathbb{Z}_n^*| = 4 \varphi(n)$ as claimed. \hfill $\square$

**Remark 3.8.** Let $\langle \rho, \mu \rangle$ be the subgroup of $G_n$ generated by $\rho$ and $\mu$. One can see that $\langle \rho, \mu \rangle = \langle \xi_1, \rho, \mu, \rho \mu, \xi_1^{-1}, \xi_1^{-1} \rho, \xi_1^{-1} \mu, \xi_1^{-1} \rho \mu \rangle$ is isomorphic to the dihedral group $D_4 = \langle r, s \mid r^4 = f^2 = (rf)^2 = 1 \rangle$, with $r$ corresponding to $\rho \mu$ or $\mu \rho$, and $f$ corresponding to any of $\rho, \mu, \rho \mu$, or $\mu \rho \mu$. The mapping $h : \mathbb{Z}_n^* \times D_4 \to G_n$ defined by

$$h(a, r^i f^j) = \xi_a (\rho \mu)^i \rho^j, \quad \text{for } i \in \{0, 1, 2, 3\}, \ j \in \{0, 1\},$$

is a group epimorphism with kernel $C_2 = \langle(-1, r^2)\rangle$, hence by the first isomorphism theorem for groups, $G_n \cong (\mathbb{Z}_n^* \times D_4)/C_2$.

The elements of $G_n$ are permutations of $\mathbb{Z}_n \times \mathbb{Z}_n$, hence the group $G_n$ acts naturally on $\mathbb{Z}_n \times \mathbb{Z}_n$. We denote this action by $\alpha_n$. In the next lemma we show how to count the isomorphism classes in a set $\mathcal{K}_n$ of $I$-graphs on $2n$ vertices, by counting the orbits of $\alpha_n$ on an appropriate subset $K_n \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$.

**Lemma 3.9.** Let $\mathcal{K}_n \subseteq \{I(n, j, k); \ j, k \in \mathbb{Z}_n^*\}$ be a set of $I$-graphs closed under isomorphism. Let $K_n \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$ satisfy

$$K_n \cap (\mathbb{Z}_n^* \times \mathbb{Z}_n^*) = \{(j, k); I(n, j, k) \in \mathcal{K}_n\},$$

and $g(K_n) = K_n$ for all $g \in G_n$. Then the restriction of $G_n$ to $K_n$,

$$G|_{K_n} := \{g|_{K_n}; g \in G_n\},$$

is a subgroup of $S(K_n)$, so let $\alpha(K_n)$ be the action of $G|_{K_n}$ on $K_n$. Write

$$\nu_0(K_n) = |\{\eta \in K_n/\sim_{\alpha(K_n)}; \eta \not\subseteq \mathbb{Z}_n^* \times \mathbb{Z}_n^*\}|.$$

Then

$$|\mathcal{K}_n/\sim| = |K_n/\sim_{\alpha(K_n)}| - \nu_0(K_n) \quad (3.3)$$

where $\sim$ denotes graph isomorphism.
Proof. Let us write \( K'_n = \{(j, k) \in \mathbb{Z}_n' \times \mathbb{Z}_n' : I(n, j, k) \in \mathcal{K}_n\} \). Note that for any \((j, k), (j', k') \in \mathbb{Z}_n' \times \mathbb{Z}_n'\) we have, by Theorem 1.1 and Proposition 3.7,

\[
I(n, j, k) \simeq I(n, j', k')
\]

\[
\iff \exists a \in \mathbb{Z}_n^* : \{j', k'\} \in \{\{aj, ak\}, \{aj, -ak\}\}
\]

\[
\iff \exists a \in \mathbb{Z}_n^* : \{j', k'\} \in \{(aj, ak), (ak, aj), (aj, -ak), (-ak, aj)\}
\]

\[
\iff \exists a \in \mathbb{Z}_n^* : \{j', k'\} \in \{\xi_a(j, k), \xi_a \rho(j, k), \xi_a \mu(j, k), \xi_a \rho \mu(j, k)\}
\]

\[
\iff \exists g \in G_n : (j', k') = g(j, k)
\]

(3.4)

where all the arithmetic is done modulo \( n \).

Let \((j, k) \in K'_n\) and \((j', k') = g(j, k)\) for some \( g \in G_n \). Then \( I(n, j, k) \in \mathcal{K}_n\), and \( I(n, j, k) \simeq I(n, j', k')\) by (3.4), hence \( I(n, j', k') \in \mathcal{K}_n\) and \((j', k') \in K'_n\). It follows that \( g(K'_n) = K'_n\) for all \( g \in G_n\), so \( G|_{K'_n}\) is a subgroup of \( S(K'_n)\). Let \( \alpha(K'_n)\) be the action of \( G|_{K'_n}\) on \( K'_n\). By Theorem 1.1, the mapping

\[ f : [I(n, j, k)] \mapsto [(j, k)] \]

from \( \mathcal{K}_n/\simeq \) to \( K'_n/\sim_{\alpha(K'_n)}\) is well defined and injective. Obviously it is also surjective, hence

\[
|\mathcal{K}_n/\simeq| = |K'_n/\sim_{\alpha(K'_n)}|.
\]

(3.5)

We claim that for any orbit \( \eta \in K_n/\sim_{\alpha(K_n)}\), either \( \eta \subseteq \mathbb{Z}_n' \times \mathbb{Z}_n'\) or \( \eta \subseteq (\mathbb{Z}_n \times \mathbb{Z}_n) \setminus (\mathbb{Z}_n' \times \mathbb{Z}_n')\). To prove this, assume that \( \eta \not\subseteq \mathbb{Z}_n' \times \mathbb{Z}_n'\). Then \((0, k) \in \eta\) or \((n/2, k) \in \eta\) for some \( k \in \mathbb{Z}_n\) (the latter only if \( n \) is even). Hence for any \((j', k') \in \eta\), there is a \( g \in G_n\) such that \((j', k') \in \{g(0, k), g(n/2, k)\}\). From Proposition 3.7 it follows that there are \( a, b, c \in \mathbb{Z}_n\) such that \(\{j', k'\} \in \{\{0, ak\}, \{bn/2, ck\}\}\). If \( n \) is even then \( b \) is odd, hence \( n | n(b - 1)/2\) and \( bn/2 \equiv n/2 \mod n\), implying that \(\{j', k'\} \in \{\{0, ak\}, \{n/2, ck\}\}\) for some \( a, c \in \mathbb{Z}_n\). We conclude that \( \eta \subseteq (\mathbb{Z}_n \times \mathbb{Z}_n) \setminus (\mathbb{Z}_n' \times \mathbb{Z}_n')\) which proves the claim.

It follows that every orbit of \( \alpha(K'_n)\) is an orbit of \( \alpha(K_n)\), and every orbit of \( \alpha(K_n)\) is either an orbit of \( \alpha(K'_n)\) or is contained in \((\mathbb{Z}_n \times \mathbb{Z}_n) \setminus (\mathbb{Z}_n' \times \mathbb{Z}_n')\). Hence

\[
|K_n/\sim_{\alpha(K_n)}| = |K'_n/\sim_{\alpha(K'_n)}| + \nu_0(K_n),
\]

which, together with (3.5), completes the proof. \(\square\)

In the rest of the paper we proceed as follows. For each of the (five) sets \( \mathcal{K}_n\) of I-graphs whose isomorphism classes we wish to enumerate, we select an appropriate set \( K_n \subseteq \mathbb{Z}_n \times \mathbb{Z}_n\), and check that the assumptions of Lemma 3.9 are satisfied. Then we count the orbits of \( \alpha(K_n)\) by means of Lemma 3.1, which is tantamount to computing the average number of fixed points of the elements \( g \in G|_{K_n}\). This is done by counting the fixed points of \( g \) in four steps, corresponding to the four possible types of \( g\), namely \( \xi_a\), \( \xi_a \mu\), \( \xi_a \rho\) and \( \xi_a \rho \mu\) (with \( a \in \mathbb{Z}_n^*\)). Finally we compute \( \nu_0(K_n)\) by counting those orbits of \( \alpha(K_n)\) that contain an element of the form \((0, k)\) or \((n/2, k)\), and use (3.3).

To simplify notation, we write \( G_n \) for \( G|_{K_n}\) and \( \alpha_n \) for \( \alpha(K_n)\) in the sequel. This causes no confusion, since in each of the five cases considered it is straightforward to verify that \( G|_{K_n} \simeq G_n\).
3.2 I-graphs

Let $K_n$ be the set of all I-graphs on $2n$ vertices, and $K_n := \mathbb{Z}_n \times \mathbb{Z}_n$.

Proposition 3.10.

$$|\mathbb{Z}_n \times \mathbb{Z}_n/\sim_{\alpha_n}| = \frac{1}{4\varphi(n)} \sum_{i=1}^{4} \sum_{a \in \mathbb{Z}_n^*} f_i(a, n)$$

where

$$f_1(a, n) = \gcd(n, a - 1)^2,$$
$$f_2(a, n) = \gcd(n, a - 1) \gcd(n, a + 1),$$
$$f_3(a, n) = \gcd(n, a^2 - 1),$$
$$f_4(a, n) = \gcd(n, a^2 + 1).$$

Proof. We use Lemma 3.1. The fixed points of $\xi_a$ are those pairs $(j, k)$ which satisfy $aj \equiv j \pmod{n}$ and $ak \equiv k \pmod{n}$. As in the proof of Proposition 3.2 we see that there are $d = \gcd(n, a - 1)$ such $j$’s, and $d$ such $k$’s, hence $d^2$ such pairs. The number of fixed points of all $\xi_a$ is thus $\sum_{a \in \mathbb{Z}_n^*} f_1(a, n)$.

The fixed points of $\xi_{a\mu}$ are those pairs $(j, k)$ which satisfy $aj \equiv j \pmod{n}$ and $-ak \equiv k \pmod{n}$. There are $\gcd(n, a - 1)$ such $j$’s, and $\gcd(n, a + 1)$ such $k$’s, hence the number of fixed points of all $\xi_{a\mu}$ is $\sum_{a \in \mathbb{Z}_n^*} f_2(a, n)$.

The fixed points of $\xi_{a\rho}$ are those pairs $(j, k)$ which satisfy $ak \equiv j \pmod{n}$ and $aj \equiv k \pmod{n}$. Hence $a^2k \equiv k \pmod{n}$, and for any such $k$, we must take $j \equiv ak \pmod{n}$. There are $\gcd(n, a^2 - 1)$ such $k$’s, hence the number of fixed points of all $\xi_{a\rho}$ is $\sum_{a \in \mathbb{Z}_n^*} f_3(a, n)$.

The fixed points of $\xi_{a\rho\mu}$ are those pairs $(j, k)$ which satisfy $-ak \equiv j \pmod{n}$ and $aj \equiv k \pmod{n}$. Hence $-a^2k \equiv k \pmod{n}$, and for any such $k$, we must take $j \equiv -ak \pmod{n}$. There are $\gcd(n, a^2 + 1)$ such $k$’s, hence the number of fixed points of all $\xi_{a\rho\mu}$ is $\sum_{a \in \mathbb{Z}_n^*} f_4(a, n)$.

Since $|G_n| = 4\varphi(n)$, the assertion follows.

Now we wish to evaluate the sum appearing in Proposition 3.10 in closed form, given the prime factorization of $n$. We do this by splitting this double sum into four single sums corresponding to $i = 1, 2, 3, 4$, evaluating each of them in the case when $n$ is a prime power, and showing that they are multiplicative.

Lemma 3.11. For $i = 1, 2, 3, 4$, let

$$g_i(n) = \frac{1}{\varphi(n)} \sum_{a \in \mathbb{Z}_n^*} f_i(a, n)$$

where $f_i(a, n)$ are as in Proposition 3.10. If $p$ is a prime and $k \geq 1$, then $g_i(p^k)$ are as given in equations (2.2) – (2.5).
Proof. Let \( x, r \in \mathbb{Z} \) with \( \gcd(r, p) = 1 \). Denote
\[
\nu_p(x) = \max \{ i \in \mathbb{N}; p^i \mid x \},
\]
\[
M_{k,j}^{(r)}(p) = \{ x \in \mathbb{Z}_{p^k}^* - r; \nu_p(x) \geq j \}, \text{ for } 1 \leq j \leq k,
\]
\[
N_{k,j}^{(r)}(p) = \{ x \in \mathbb{Z}_{p^k}^* - r; \nu_p(x) = j \}, \text{ for } 0 \leq j \leq k - 1.
\]

The elements of \((\mathbb{Z}_{p^k} \setminus \mathbb{Z}_{p^k}^*) - r\) are not divisible by \( p \), hence it follows for \( j \geq 1 \) that \( M_{k,j}^{(r)}(p) = \{ x \in \mathbb{Z}_{p^k}^* - r; \nu_p(x) \geq j \} \). This is the set of all multiples of \( p^j \) in a set of \( p^k \) consecutive integers, therefore Lemma 3.5 (iii) implies that \(|M_{k,j}^{(r)}(p)| = p^{k-j}\) for \( 1 \leq j \leq k \) and for all \( r \) such that \( \gcd(r, p) = 1 \). Consequently
\[
|N_{k,j}^{(r)}(p)| = |M_{k,j}^{(r)}(p)| - |M_{k,j+1}^{(r)}(p)| = p^{k-j} - p^{k-j-1} \quad \text{for } 1 \leq j \leq k - 1,
\]
\[
|N_{k,0}^{(r)}(p)| = |\mathbb{Z}_{p^k}^* - r| - |M_{k,1}^{(r)}(p)| = \varphi(p^k) - p^{k-1} = p^k - 2p^{k-1}.
\]

It follows that for any \( s \in \mathbb{N} \) we have
\[
\sum_{a \in \mathbb{Z}_{p^k}^*} \gcd(p^k, a - r)^s = \sum_{j=0}^{k-1} |N_{k,j}^{(r)}(p)|p^{sj} + |M_{k,k}^{(r)}(p)|p^{sk}
\]
\[
= p^k - 2p^{k-1} + p^k \sum_{j=1}^{k-1} (p^{(s-1)j} - p^{(s-1)j-1}) + p^{sk} \quad (3.6)
\]
which for \( s = 1 \) turns into
\[
\sum_{a \in \mathbb{Z}_{p^k}^*} \gcd(p^k, a - r) = (k + 1) \varphi(p^k). \quad (3.7)
\]

Now we compute \( g_i(p^k) \) for \( i = 1, 2, 3, 4 \).

(i) By (3.6) with \( r = 1 \) and \( s = 2 \) we have
\[
g_1(p^k) \varphi(p^k) = \sum_{a \in \mathbb{Z}_{p^k}^*} \gcd(p^k, a - 1)^2 = p^{k-1}((p + 1)p^k - 2),
\]
and so \( g_1(p^k) = ((p + 1)p^k - 2)/(p - 1) \) as claimed in (2.2).
(ii) For $p = 2$ and $k \geq 2$ we find, using (3.7) in the next-to-last step, that

\begin{align*}
g_2(2^k) \varphi(2^k) & = \sum_{a \in \mathbb{Z}_{2^k}^*} \gcd(2^k, a - 1) \gcd(2^k, a + 1) \\
& = \sum_{j=0}^{2^{k-1}-1} \gcd(2^k, 2j) \gcd(2^k, 2j + 2) \\
& = 4 \sum_{j=0}^{2^{k-1}-1} \gcd(2^{k-1}, j) \gcd(2^{k-1}, j + 1) \\
& = 4 \sum_{i=0}^{2^{k-2}-1} \gcd(2^{k-1}, 2i) \gcd(2^{k-1}, 2i + 1) \\
& \quad + 4 \sum_{i=0}^{2^{k-2}-1} \gcd(2^{k-1}, 2i + 1) \gcd(2^{k-1}, 2i + 2) \\
& = 4 \sum_{i=0}^{2^{k-2}-1} \gcd(2^{k-1}, 2i) + 4 \sum_{i=0}^{2^{k-2}-1} \gcd(2^{k-1}, 2i + 2) \\
& = 8 \sum_{i=0}^{2^{k-2}-1} \gcd(2^{k-1}, 2i) = 8 \sum_{a \in \mathbb{Z}_{2^{k-1}}^*} \gcd(2^{k-1}, a - 1) \\
& = 8k \varphi(2^{k-1}) = 4k \varphi(2^k), \\
\end{align*}

as claimed in (2.3). The case $k = 1$ is easily verified directly.

If $p > 2$ then at most one of $a - 1$, $a + 1$ is divisible by $p$. Hence we find, using (3.7), that

\begin{align*}
g_2(p^k) \varphi(p^k) & = \sum_{a \in \mathbb{Z}_{p^k}^*} \gcd(p^k, a - 1) \gcd(p^k, a + 1) \\
& = \sum_{a \in \mathbb{Z}_{p^k}^*} \gcd(p^k, a - 1) + \sum_{a \in \mathbb{Z}_{p^k}^*} \gcd(p^k, a + 1) - \sum_{a \in \mathbb{Z}_{p^k}^*} 1 \\
& = 2(k + 1) \varphi(p^k) - \varphi(p^k) = (2k + 1) \varphi(p^k)
\end{align*}

and (2.3) follows.
(iii) For \( p = 2 \) and \( k \geq 2 \) we obtain

\[
g_3(2^k) \varphi(2^k) = \sum_{a \in \mathbb{Z}_{2^k}^*} \gcd(2^k, a^2 - 1) = \sum_{j=0}^{2^k-1} \gcd(2^k, (2j + 1)^2 - 1)
\]

\[
= 4 \sum_{j=0}^{2^k-1} \gcd(2^{k-2}, j(j + 1))
\]

\[
= 4 \sum_{j=0}^{2^{k-2}-1} \gcd(2^{k-2}, j) \gcd(2^{k-2}, j + 1)
\]

\[
+ 4 \sum_{j=0}^{2^{k-2}-1} \gcd(2^{k-2}, j + 2^{k-2}) \gcd(2^{k-2}, j + 1 + 2^{k-2})
\]

\[
= 8 \sum_{j=0}^{2^{k-2}-1} \gcd(2^{k-2}, j) \gcd(2^{k-2}, j + 1)
\]

\[
= 8(k - 1) \varphi(2^{k-1}) = 4(k - 1) \varphi(2^k)
\]

by (3.8) and (3.9). The case \( k = 1 \) is easily verified directly.

If \( p > 2 \) then at most one of \( a - 1, a + 1 \) is divisible by \( p \). It follows that \( \gcd(p^k, a^2 - 1) = \gcd(p^k, a - 1) \gcd(p^k, a + 1) \), and so \( g_3(p^k) = g_2(p^k) = 2k + 1 \), proving (2.4).

(iv) For \( p = 2 \) we have

\[
g_4(2^k) \varphi(2^k) = \sum_{a \in \mathbb{Z}_{2^k}^*} \gcd(2^k, a^2 + 1) = \sum_{j=0}^{2^k-1} \gcd(2^k, (2j + 1)^2 + 1)
\]

\[
= 2 \sum_{j=0}^{2^k-1} \gcd(2^{k-1}, 2j^2 + 2j + 1) = 2 \cdot 2^{k-1} = 2 \varphi(2^k).
\]

Assume that \( p \equiv 1 \pmod{4} \). Then \(-1\) is a quadratic residue modulo \( p^k \), so there is an \( r \in \mathbb{Z} \) such that \( r^2 \equiv -1 \pmod{p^k} \). By Lemma 3.5 (i), \( \gcd(p^k, a^2 + 1) = \gcd(p^k, a^2 - r^2) \), hence

\[
g_4(p^k) \varphi(p^k) = \sum_{a \in \mathbb{Z}_{p^k}^*} \gcd(p^k, a^2 + 1) = \sum_{a \in \mathbb{Z}_{p^k}^*} \gcd(p^k, a^2 - r^2)
\]

\[
= \sum_{a \in \mathbb{Z}_{p^k}^*} \gcd(p^k, (a - r)(a + r)).
\]

If \( p \mid a - r \) and \( p \mid a + r \) then \( p \mid 2a \) which is false, since \( p \) is odd and \( a \in \mathbb{Z}_{p^k}^* \). Hence at most one of \( a - r, a + r \) is divisible by \( p \). Now by the same argument as in (ii) we find that \( g_4(p^k) \varphi(p^k) = (2k + 1) \varphi(p^k) \), hence \( g_4(p^k) = 2k + 1 \).
Finally, let \( p \equiv 3 \pmod{4} \). Then \(-1\) is a quadratic nonresidue modulo \( p \), hence \( \gcd(p^k, a^2 + 1) = 1 \) for all \( a \). It follows that
\[
g_4(p^k) \varphi(p^k) = \sum_{a \in \mathbb{Z}_p^*} \gcd(p^k, a^2 + 1) = \varphi(p^k)
\]
and so \( g_4(p^k) = 1 \), proving (2.5).

It remains to show that \( g_1(n), g_2(n), g_3(n), g_4(n) \) are multiplicative.

**Lemma 3.12.** Let
\[
g(n) = \sum_{a \in \mathbb{Z}_n^*} \prod_{k=1}^{r} \gcd(n, P_k(a))
\]
where \( P_1(x), P_2(x), \ldots, P_r(x) \) are polynomials in \( x \) with integer coefficients. Then \( g(n) \) is a multiplicative arithmetic function.

**Proof.** Let \( n = n_1 n_2 \) where \( \gcd(n_1, n_2) = 1 \). We need to show that \( g(n) = g(n_1)g(n_2) \).

For \( a \in \mathbb{Z}_n \), let \( a_1 \in \mathbb{Z}_{n_1} \) and \( a_2 \in \mathbb{Z}_{n_2} \) be such that
\[
a \equiv a_1 \pmod{n_1}, \quad a \equiv a_2 \pmod{n_2}.
\]

By the Chinese Remainder Theorem, the mapping
\[
f : \ a \mapsto (a_1, a_2)
\]
is a bijection from \( \mathbb{Z}_n \) to \( \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \). By Lemma 3.5 (i) and (ii), \( \gcd(n_1 n_2, a) = 1 \) iff \( \gcd(n_1, a) = \gcd(n_2, a) = 1 \) iff \( \gcd(n_1, a_1) = \gcd(n_2, a_2) = 1 \), therefore \( f \) restricted to \( \mathbb{Z}_n^* \) is a bijection from \( \mathbb{Z}_n^* \) to \( \mathbb{Z}_{n_1}^* \times \mathbb{Z}_{n_2}^* \). Also, \( P_k(a) \equiv P_k(a_i) \pmod{n_i} \) for \( i = 1, 2 \), hence by Lemma 3.5 (i) and (ii),
\[
\gcd(n_1 n_2, P_k(a)) = \gcd(n_1, P_k(a_1)) \gcd(n_2, P_k(a_2)) = \gcd(n_1, P_k(a)) \gcd(n_2, P_k(a)).
\]

It follows that
\[
g(n_1 n_2) = \sum_{(a_1, a_2) \in \mathbb{Z}_{n_1}^* \times \mathbb{Z}_{n_2}^*} \prod_{k=1}^{r} \gcd(n_1, P_k(a_1)) \gcd(n_2, P_k(a_2)) = \sum_{a_1 \in \mathbb{Z}_{n_1}} \prod_{k=1}^{r} \gcd(n_1, P_k(a_1)) \sum_{a_2 \in \mathbb{Z}_{n_2}} \prod_{k=1}^{r} \gcd(n_2, P_k(a_2)) = g(n_1)g(n_2),
\]
proving multiplicativity of \( g(n) \). 

**Proof of Theorem 2.1:**

Clearly \( I(n) = |\mathcal{K}_n| \simeq |, \) and the assumptions of Lemma 3.9 are satisfied. We still need to compute \( \nu_0(\mathbb{Z}_n \times \mathbb{Z}_n) \). From Corollary 3.4 (iii) it follows that the set \( U_n := (\{0\} \times \mathbb{Z}_n) \cup (\mathbb{Z}_n \times \{0\}) \) equals the union of \( \tau(n) \) orbits with representatives \((0, k)\) where \( k | n \) (with \( k = n \) replaced by 0). So if \( n \) is odd, \( \nu_0(\mathbb{Z}_n \times \mathbb{Z}_n) = \tau(n) \). If \( n \) is even, the set \( V_n := (\{n/2\} \times \mathbb{Z}_n) \cup (\mathbb{Z}_n \times \{n/2\}) \) equals the union of \( \tau(n) \) orbits with representatives \((n/2, k)\) where \( k | n \) (with \( n \) replaced by 0). The two sets \( U_n \) and \( V_n \) share the orbit containing \((n/2, 0)\), hence in this case \( \nu_0(\mathbb{Z}_n \times \mathbb{Z}_n) = 2\tau(n) - 1 \). Equation (2.1) now follows by Lemma 3.9, using Proposition 3.10, Lemma 3.11 and Lemma 3.12. 

\( \square \)
3.3 Generalized Petersen graphs

Let $K_n$ be the set of all generalized Petersen graphs on $2n$ vertices, and

$$K_n := \mathbb{Z}^*_n \times \mathbb{Z}_n \cup \mathbb{Z}_n \times \mathbb{Z}^*_n.$$ 

**Proposition 3.13.**

$$|K_n/\sim_{\alpha_n}| = \frac{1}{4}(2n - \varphi(n) + 2 \gcd(n, 2) + r(n) + s(n)) \quad (3.10)$$

**Proof.** We use Lemma 3.1. Assume that $(j, k) \in K_n$ is fixed by some $g \in G_n$.

a) If $g = \xi_a$ then $(aj, ak) = (j, k)$. Since $\{j, k\} \cap \mathbb{Z}^*_n \neq \emptyset$, it follows that $a \equiv 1 \pmod{n}$. So $\sum_{a \in \mathbb{Z}_n} \text{fix}_{\alpha_n}(\xi_a) = \text{fix}_{\alpha_n}(\xi_1) = |K_n| = n^2 - (n - \varphi(n))^2 = \varphi(n)(2n - \varphi(n)).$

b) If $g = \xi_a \rho$ then $(ak, aj) = (j, k)$. In this case $a^2j \equiv j \pmod{n}$ and $a^2k \equiv k \pmod{n}$, so $a^2 \equiv 1 \pmod{n}$. Thus $\sum_{a \in \mathbb{Z}_n} \text{fix}_{\alpha_n}(\xi_a \rho) = r(n)\varphi(n)$.

c) If $g = \xi_a \mu$ then $(ak, aj) = (j, k)$. In this case $a^2j \equiv j \pmod{n}$ and $a^2k \equiv k \pmod{n}$, so $a^2 \equiv 1 \pmod{n}$. Thus $\sum_{a \in \mathbb{Z}_n} \text{fix}_{\alpha_n}(\xi_a \mu) = s(n)\varphi(n)$.

Equation (3.10) now follows from Lemma 3.1.

**Proof of Theorem 2.2:**

Clearly $P(n) = |K_n/\sim|$. It follows from Theorem 1.1 that $I(n, j, k)$ is isomorphic to a generalized Petersen graph if and only if $j \in \mathbb{Z}^*_n$ or $k \in \mathbb{Z}^*_n$, hence the assumptions of Lemma 3.9 are satisfied. We still need to compute $\nu_0(K_n)$, the number of orbits containing pairs of the form $(0, k)$ or $(n/2, k)$ with $k \in \mathbb{Z}^*_n$. There are two such orbits if $n$ is even, and one if $n$ is odd, hence $\nu_0(K_n) = \gcd(n, 2)$. Equation (2.6) now follows by Lemma 3.9, using Proposition 3.13.

3.4 Connected I-graphs

Let $K_n$ be the set of all connected I-graphs on $2n$ vertices, and

$$K_n := \{(j, k) \in \mathbb{Z}_n \times \mathbb{Z}_n; \gcd(n, j, k) = 1\}.$$

**Proposition 3.14.**

$$|K_n/\sim_{\alpha_n}| = \frac{1}{4} \left( \frac{J_2(n)}{\varphi(n)} + r(n) + s(n) + t(n) \right) \quad (3.11)$$

where $t(n) = t_1(n) + t_2(n)$ is given in (2.8).

**Proof.** We use Lemma 3.1. Assume that $(j, k) \in K_n$ is fixed by some $g \in G_n$.

a) If $g = \xi_a$ then $(aj, ak) = (j, k)$. Let $d = \gcd(n, a-1)$, $n = n' d$ and $a - 1 = a' d$. As in the proof of Proposition 3.2, we see that $n' | j$ and $n' | k$. Since $n' | n$ as well, it follows
that $n' = 1$ and so $n \mid a - 1$, which is only possible if $a = 1$. Thus $\xi_a$ has no fixed points unless $a = 1$. As $\xi_1$ fixes all points in $K_n$, we have

$$\sum_{a \in \mathbb{Z}_n^*} \text{fix}_{\alpha_n}(\xi_a) = \text{fix}_{\alpha_n}(\xi_1) = |K_n| = J_2(n).$$

b) If $g = \xi_a \mu$ then $(aj, -ak) = (j, k)$. Denote $n_j = \gcd(n, j)$ and $n_k = \gcd(n, k)$. Any common divisor of $n_j$ and $n_k$ is a common divisor of $n, j, k$, hence $n_j \perp n_k$ and $n_j n_k \mid n$. Denote $n_0 = n/(n_j n_k), j' = j/n_j, k' = k/n_k$. Then

$$n = n_0 n_j n_k, j' \in \mathbb{Z}_{n_0 n_k}^*, k' \in \mathbb{Z}_{n_0 n_j}^*.$$

From $aj \equiv j \pmod{n}$ it follows that $n_0 n_j \mid (a - 1)j'$, hence $n_0 n_j \mid a - 1$. From $ak \equiv -k \pmod{n}$ it follows that $n_0 n_j \mid (a + 1)k'$, hence $n_0 n_j \mid a + 1$. Therefore $n_0 \mid 2$, and so $n_0 \in \{1, 2\}$ and $\varphi(n_0) = 1$.

We claim that for each pair $(j, k)$ where $j = j'n_j, k = k'n_k, n = n_0 n_j n_k, n_0 \in \{1, 2\}, n_j \perp n_k, j' \in \mathbb{Z}_{n_0 n_j}^*$ and $k' \in \mathbb{Z}_{n_0 n_k}^*$, there is a unique $a \in \mathbb{Z}_n^*$ such that $aj \equiv j \pmod{n}$ and $ak \equiv -k \pmod{n}$. Indeed, let $n = \prod_{i=1}^m p_i^{e_i}$ be the prime factorization of $n$ (i.e., $p_1, p_2, \ldots, p_m$ are distinct primes and $e_i \geq 1$ for $i = 1, 2, \ldots, m$). Define $a \in \mathbb{Z}$ by requiring that for each $i \in \{1, 2, \ldots, m\}$,

$$a \equiv -1 \pmod{p_i^{e_i}} \text{ if } p_i^{e_i} \mid n_0 n_j, \quad a \equiv 1 \pmod{p_i^{e_i}} \text{ if } p_i^{e_i} \mid n_0 n_k.$$

At least one of $p_i^{e_i} \mid n_0 n_j$ and $p_i^{e_i} \mid n_0 n_k$ holds for each $i \in \{1, 2, \ldots, m\}$, and both hold only if $p_i^{e_i} = n_0 = 2$, hence these requirements are consistent, and by the Chinese Remainder Theorem, there is a unique $a \in \mathbb{Z}_n$ which satisfies them. In fact, $a^2 \equiv 1 \pmod{p_i^{e_i}}$ for $i = 1, 2, \ldots, m$, hence $a^2 \equiv 1 \pmod{n}$, and so $a \in \mathbb{Z}_n^*$. Note that $a$ is odd if $n_0 = 2$, therefore $n_0 \mid a - 1$ and $n_0 \mid a + 1$.

If $p_i^{e_i} \mid n_0 n_j$ then $p_i^{e_i} \mid n_0 j \mid (a - 1)j$. Also, $a \equiv -1 \pmod{p_i^{e_i}}$, so $p_i^{e_i} \mid (a + 1)k$.

If $p_i^{e_i} \mid n_0 n_k$ then $p_i^{e_i} \mid n_0 k \mid (a + 1)k$. Also, $a \equiv 1 \pmod{p_i^{e_i}}$, so $p_i^{e_i} \mid (a - 1)j$.

In either case, $p_i^{e_i} \mid (a - 1)j$ and $p_i^{e_i} \mid (a + 1)k$. As this holds for all $i \in \{1, 2, \ldots, m\}$, it follows that $n \mid (a - 1)j$ and $n \mid (a + 1)k$, hence $aj \equiv j \pmod{n}$ and $ak \equiv -k \pmod{n}$ as claimed.

Thus to construct $(j, k) \in K_n$ which is fixed by some $\xi_a \mu$, first select $n_0, n_j, n_k, j', k' \in \mathbb{Z}_n$ such that $n_0 \in \{1, 2\}, n_j \perp n_k, n = n_0 n_j n_k, j' \in \mathbb{Z}_{n_0 n_j}^*$ and $k' \in \mathbb{Z}_{n_0 n_k}^*$, then take $j = j'n_j, k = k'n_k$. This can be done in

$$\sum_{n_0 \in \{1, 2\}, n_j \perp n_k, n = n_0 n_j n_k} \varphi(n_0 n_k) \varphi(n_0 n_j)$$

ways. W.l.o.g. assume that $n_k$ is odd. Then $\varphi(n_0 n_k) \varphi(n_0 n_j) = \varphi(n_0) \varphi(n_k) \varphi(n_0 n_j) = \varphi(n_k) \varphi(n_0 n_j) = \varphi(n_0 n_j) \varphi(n) \varphi(n_j) = \varphi(n), \quad \text{hence}

$$\sum_{a \in \mathbb{Z}_n^*} \text{fix}_{\alpha_a}(\xi_a \mu) = \varphi(n)(t_1(n) + t_2(n))$$

where $t_{n_0}(n) = |\{(n_j, n_k); n_j \perp n_k, n = n_0 n_j n_k\}|$. Clearly, $t_1(n) = 2^\omega(n)$ and

$$t_2(n) = \begin{cases} 2^{\omega(n/2)}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$
c) If \( g = \xi_a \rho \) then \((ak, aj) = (j, k)\). In this case \( \gcd(n, j, aj) = \gcd(n, j, k) = 1 \) by Lemma 3.5 (i), and \( a^2 j \equiv j \pmod n \). It follows that \( j \in \mathbb{Z}_n^* \) and \( a^2 \equiv 1 \pmod n \). Since \( k \equiv aj \pmod n \) is determined by the choice of \( j \in \mathbb{Z}_n^* \), we have \( \sum_{a \in \mathbb{Z}_n^*} \fix_{\alpha_n}(\xi_a \rho) = r(n)\varphi(n) \).

d) If \( g = \xi_a \rho \mu \) then \((-ak, aj) = (j, k)\). In this case \( \gcd(n, j, aj) = \gcd(n, j, k) = 1 \) by Lemma 3.5 (i), and \( a^2 j \equiv -j \pmod n \). It follows that \( j \in \mathbb{Z}_n^* \) and \( a^2 \equiv -1 \pmod n \). Since \( k \equiv aj \pmod n \) is determined by the choice of \( j \in \mathbb{Z}_n^* \), we have \( \sum_{a \in \mathbb{Z}_n^*} \fix_{\alpha_n}(\xi_a \rho \mu) = s(n)\varphi(n) \).

Equation (3.11) now follows from Lemma 3.1.

\( \square \)

**Proof of Theorem 2.3:**

Clearly \( I_c(n) = |K_n/\simeq| \). It follows from Theorem 1.2 that the assumptions of Lemma 3.9 are satisfied. We still need to compute \( \nu_0(K_n) \), the number of orbits containing pairs of the form \((0, k)\) or \((n/2, k)\) with \( k \in \mathbb{Z}_n^* \).

If \((0, k) \in K_n\) then \( \gcd(n, k) = \gcd(n, 0, k) = 1 \), hence \( k \in \mathbb{Z}_n^* \). It follows that all such pairs belong to a single orbit of \( \alpha_n \).

Assume that \( n \equiv 0 \pmod 4 \). If \((n/2, k) \in K_n\) then \( \gcd(n, n/2, k) = 1 \). Since in this case \( \gcd(n, n/2, k) = 1 \) iff \( \gcd(n, k) = 1 \), it follows that \( k \in \mathbb{Z}_n^* \). For any \( a \in \mathbb{Z}_n^* \), we have \( a(n/2) \equiv n/2 \pmod n \), hence we conclude again that all such pairs belong to a single orbit of \( \alpha_n \).

Assume that \( n \equiv 2 \pmod 4 \). If \((n/2, k) \in K_n\) then \( \gcd(n, n/2, k) = 1 \). In this case it is straightforward to see that \( \gcd(n, n/2, k) = 1 \) iff \( k = 2^ja \) for some \( j \geq 0 \) and \( a \in \mathbb{Z}_n^* \). All the pairs \((n/2, a)\) with \( a \in \mathbb{Z}_n^* \) clearly belong to a single orbit of \( \alpha_n \). Now we claim that \( 4\mathbb{Z}_n^* = 2\mathbb{Z}_n^* \). Indeed, let \( q = n/2 \) and \( a \in \mathbb{Z}_n^* \). Then \( \gcd(2a + q, n) = 1 \) and \( 4a \equiv 2(2a + q) \pmod n \), proving that \( 4\mathbb{Z}_n^* \subseteq 2\mathbb{Z}_n^* \). Conversely, if \( q \equiv 1 \pmod 4 \) then \( \gcd((q + 1)/2, n) = 1 \) and \( 2a \equiv 4a(q + 1)/2 \pmod n \). If \( q \equiv 3 \pmod 4 \) then \( \gcd((3q + 1)/2, n) = 1 \) and \( 2a \equiv 4a((3q + 1)/2) \pmod n \), proving that \( 2\mathbb{Z}_n^* \subseteq 4\mathbb{Z}_n^* \), and also the claim. Hence all the pairs \((n/2, 2^ja)\) with \( j \geq 1 \) and \( a \in \mathbb{Z}_n^* \) also belong to a single orbit of \( \alpha_n \). On the other hand, all the pairs in the orbit of \((n/2, 1)\) have one component in \( \mathbb{Z}_n^* \), while all the pairs in the orbit of \((n/2, 2)\) have neither component in \( \mathbb{Z}_n^* \), hence these two orbits are distinct.

It follows that

\[
\nu_0(K_n) = \begin{cases} 
1, & n \equiv 1 \pmod 2, \\
2, & n \equiv 0 \pmod 4, \\
3, & n \equiv 2 \pmod 4, 
\end{cases}
\]

which together with Lemma 3.9 and Proposition 3.14 yields (2.7).

\( \square \)

### 3.5 Bipartite generalized Petersen graphs

Let \( K_n \) be the set of all bipartite generalized Petersen graphs on \( 2n \) vertices, and

\[
K_n := \mathbb{Z}_n^* \times \mathbb{Z}_n^o \cup \mathbb{Z}_n^o \times \mathbb{Z}_n^*,
\]

where \( \mathbb{Z}_n^o \) is the subset of odd elements in \( \mathbb{Z}_n \).

**Proposition 3.15.** Let \( n \) be even. Then

\[
|K_n/\sim_{\alpha_n}| = \frac{1}{4} (n - \varphi(n) + 2 ((n/2) \mod 2) + r(n) + s(n)).
\]

(3.12)
Define the functions \( f \) which together with Lemma 3.9 and Proposition 3.15 yields (2.9). (mod 4) \( \nu \) to compute Proposition 3.16.

Let odd and \( k \) or \( k \) isomorphic to a bipartite generalized Petersen graph if and only if \( j \) \( \nu \) is fixed by some \( g \in G_n \), and notice that both \( j \) and \( k \) are odd.

a) If \( g = \xi_a \) then \((a, ak) = (j, k)\). From \( \{j, k\} \cap \mathbb{Z}_n^* \neq \emptyset \) it follows that \( a \equiv 1 \mod n \). So \( \sum_{a \in \mathbb{Z}_n^*} \text{fix}_{\alpha_n}(\xi_a) = |K_n| = (n/2)^2 - (n/2 - \varphi(n))^2 = \varphi(n)(n - \varphi(n)) \).

b) If \( g = \xi_{a\mu} \) then \((a, -ak) = (j, k)\). If \( j \in \mathbb{Z}_n^* \), then \( a \equiv 1 \mod n \) and \( 2k \equiv 0 \mod n \). As \( k \) is odd, this is only possible if \( n \not\equiv 0 \mod 4 \) and \( k = n/2 \). If \( k \in \mathbb{Z}_n^* \), then \( a \equiv -1 \mod n \), \( n \not\equiv 0 \mod 4 \) and \( j = n/2 \). So \( \text{fix}_{\alpha_n}(\xi_1 \mu) = \text{fix}_{\alpha_n}(\xi_{-1} \mu) = \varphi(n)(n/2 \mod 2) \), and \( \sum_{a \in \mathbb{Z}_n^*} \text{fix}_{\alpha_n}(\xi_a \mu) = 2\varphi(n)(n/2 \mod 2) \).

c), d): As in the proof of Proposition 3.13.

Equation (3.12) now follows from Lemma 3.1. \( \square \)

Proof of Theorem 2.4:

Clearly \( P_b(n) = |K_n/\simeq| \). It follows from Theorems 1.1 and 1.3 that \( I(n, j, k) \) is isomorphic to a bipartite generalized Petersen graph if and only if \( j \in \mathbb{Z}_n^* \) and \( k \) is odd, or \( k \in \mathbb{Z}_n^* \) and \( j \) is odd, hence the assumptions of Lemma 3.9 are satisfied. We still need to compute \( \nu_0(K_n) \), the number of orbits containing pairs of the form \((n/2, k)\) with \( n/2 \) odd and \( k \in \mathbb{Z}_n^* \). There are no such orbits if \( n \equiv 0 \mod 4 \), and one such orbit if \( n \equiv 2 \mod 4 \). Hence

\[ \nu_0(K_n) = (n/2) \mod 2, \]

which together with Lemma 3.9 and Proposition 3.15 yields (2.9). \( \square \)

3.6 Bipartite connected \( I \)-graphs

Let \( \mathcal{K}_n \) be the set of all bipartite connected \( I \)-graphs on \( 2n \) vertices, and

\[ K_n := \{(j, k) \in \mathbb{Z}_n \times \mathbb{Z}_n; \gcd(n, j, k) = 1, \ j, k \text{ odd}\}. \]

Proposition 3.16. Let \( n \) be even. Then

\[ |K_n/\sim_{\alpha_n}| = \frac{1}{4} \left( \frac{J_2(n)}{3\varphi(n)} + ((n/2) \mod 2) 2^{\omega(n/2)} + r(n) + s(n) \right). \tag{3.13} \]

Proof. We follow the proof of Proposition 3.14. Assume that \((j, k) \in K_n\) is fixed by some \( g \in G_n \).

a) If \( g = \xi_a \) then \((a, ak) = (j, k)\). As in case a) in the proof of Proposition 3.14, we see that \( a \equiv 1 \mod n \), thus \( \sum_{a \in \mathbb{Z}_n^*} \text{fix}_{\alpha_n}(\xi_a) = \text{fix}_{\alpha_n}(\xi_1) = |K_n| \). Let

\begin{align*}
U_n &:= \{(j, k) \in \mathbb{Z}_n \times \mathbb{Z}_n; \gcd(n, j, k) = 1, \ j \text{ odd, } k \text{ even}\}, \\
V_n &:= \{(j, k) \in \mathbb{Z}_n \times \mathbb{Z}_n; \gcd(n, j, k) = 1, \ j \text{ even, } k \text{ odd}\}, \\
W_n &:= \{(j, k) \in \mathbb{Z}_n \times \mathbb{Z}_n; \gcd(n, j, k) = 1\}.
\end{align*}

Define the functions \( f_n : K_n \to U_n \) and \( g_n : U_n \to K_n \) by

\begin{align*}
f_n(j, k) &:= (j, k + j) \mod n, \\
g_n(j, k) &:= (j, k - j) \mod n.
\end{align*}

Clearly \( \gcd(n, j, k) = 1 \) iff \( \gcd(n, j, k + j) = 1 \) iff \( \gcd(n, j, k - j) = 1 \). Next, for \( j, k \) odd, \( k + j \mod n \) is even, and if \( j \) is odd and \( k \) is even, then \( k - j \mod n \) is odd.
Since \( f_n(g_n(j, k)) = (j, k) = g_n(f_n(j, k)) \), we conclude that \( f_n \) and \( g_n \) are bijections, and \(|K_n| = |U_n|\). Since \( W_n = K_n \cup U_n \cup V_n, |W_n| = J_2(n) \), and \(|U_n| = |V_n|\) by symmetry, it follows that \(|K_n| = |U_n| = |V_n| = J_2(n)/3\).

b) If \( g = \xi_{a\mu} \) then \((a, -ak) = (j, k)\). As in case b) in the proof of Proposition 3.14, we see that \( n = n_0n_jn_k \) where \( n_0 \mid 2, n_j \mid j \) and \( n_k \mid k \). Since \( n \) is even while \( j \) and \( k \) are odd, it follows that \( n_0 = 2 \), hence \( \xi_{a\mu} \) has no fixed points if \( n \equiv 0 \pmod{4} \). So assume that \( n \equiv 2 \pmod{4} \). To construct \((j, k) \in K_n \) which is fixed by some (uniquely determined) \( \xi_{a\mu} \), first select \( n_j, n_k, j', k' \in \mathbb{Z}_n \) such that \( n_j \perp n_k, n = 2n_jn_k, j' \in \mathbb{Z}^*_2n_k \) and \( k' \in \mathbb{Z}^*_2n_j \), then take \( j = j'n_j, k = k' n_k \). This can be done in

\[
\sum_{n_j \perp n_k, n=2n_jn_k} \varphi(2n_k)\varphi(2n_j)
\]

ways. Since \( n_k \) and \( n_j \) are odd, \( \varphi(2n_k)\varphi(2n_j) = \varphi(n_k)\varphi(2n_j) = \varphi(2n_kn_j) = \varphi(n) \). Therefore \( \sum_{n \in \mathbb{Z}_n^*} \text{fix}_\alpha(n, \xi_{a\mu}) = \varphi(n)2^{\omega(n/2)} \) if \( n \equiv 2 \pmod{4} \). By multiplying this expression with \((n/2) \pmod{2}\) we extend its validity to all even \( n \).

c), d): As in the proof of Proposition 3.14.

Equation (3.13) now follows from Lemma 3.1.

\[ \square \]

Proof of Theorem 2.5:

Clearly \( I_{bc}(n) = |\mathcal{K}_n/\sim| \). It follows from Theorems 1.2 and 1.3 that the assumptions of Lemma 3.9 are satisfied. We still need to compute \( \nu_0(K_n) \), the number of orbits containing pairs of the form \((n/2, k)\) with \( n/2 \) and \( k \) odd and \( \gcd(n, n/2, k) = 1 \). In this case \( \gcd(n, n/2, k) = 1 \) if and only if \( \gcd(n, k) = 1 \). Therefore there are no such orbits if \( n \equiv 0 \pmod{4} \), and one such orbit if \( n \equiv 2 \pmod{4} \). Hence

\[ \nu_0(K_n) = (n/2) \pmod{2}, \]

which together with Lemma 3.9 and Proposition 3.16 yields (2.10).

\[ \square \]

4 Concluding remark

It is not difficult to see that the numbers \( I_c(n) \) and \( I(n) \) of isomorphism classes of connected I-graphs resp. all I-graphs on \( 2n \) vertices satisfy the pair of Moebius inverse relations

\[ I(n) = \sum_{d \mid n} I_c(d), \quad I_c(n) = \sum_{d \mid n} \mu(n/d)I(d) \]

(cf. [8, Sec. 3]).

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References


