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A note on acyclic number of planar graphs*

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Abstract

The acyclic number $a(G)$ of a graph G is the maximum order of an induced forest in G. The purpose of this short paper is to propose a conjecture that $a(G) \geq \left(1 - \frac{3}{2g}\right) n$ holds for every planar graph G of girth g and order n, which captures three known conjectures on the topic. In support of this conjecture, we prove a weaker result that $a(G) \geq \left(1 - \frac{3}{g}\right) n$ holds. In addition, we give a construction showing that the constant $\frac{3}{2}$ from the conjecture cannot be decreased.

Keywords: Induced forest, acyclic number, planar graph, girth.

Math. Subj. Class.: 05C10, 05C15

1 Introduction

Throughout the paper n and g, respectively, stand for the order and girth of a (finite, simple, undirected) graph G. For other standard terminology and notation of graph theory we simply refer to [5]. The *acyclic number* of G, denoted $a(G)$, is the maximum order of an induced forest in G. This parameter has been well investigated (see e.g. [1, 4, 9, 10]), and its determination is NP-hard even in the case of planar graphs [7]. In [2], Albertson and Berman proposed the following lower bound for it.

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Conjecture 1.1. *If G is a planar graph, then*

$$a(G) \geq \frac{n}{2}.$$

This conjecture has drawn much attention since it implies that every planar graph has a stable set on at least a quarter of its vertices, a fact known to be true only as a consequence of the Four Color Theorem. It holds for planar graphs of girth at least 4 as Salavatipour [10] (see also [4]) proved that $a(G) \geq \frac{17n+24}{32}$ whenever G is such a graph. The best known lower bound for $a(G)$ over the class of all planar graphs G is the inequality $a(G) \geq \frac{2n}{5}$, which can be readily deduced from the acyclic 5-colorability of planar graphs (proven by Borodin in [6]). A similar problem to Conjecture 1.1 is Conjecture 1.2 below, raised by Akiyama and Watanabe [1].

Conjecture 1.2. *If G is a bipartite planar graph, then*

$$a(G) \geq \frac{5n}{8}.$$

Motivated by the last conjecture, the existence of large induced acyclic subgraphs in sparse bipartite graphs (resp. sparse graphs) was considered by Alon et al. in [3] (resp. [4]). Inspired by the fact that the dodecahedron attains the minimum possible ratio of order to size among all connected planar graphs of girth at least 5, Kowalik et al. [8] conjectured the following.

Conjecture 1.3. *If G is a planar graph of girth $g \geq 5$, then*

$$a(G) \geq \frac{7n}{10}.$$

The main purpose of this note is to generalize Conjectures 1.1, 1.2 and 1.3 through the following.

Conjecture 1.4. *If G is a planar graph of girth g, then*

$$a(G) \geq \left(1 - \frac{3}{2g}\right) n.$$

In particular, our conjecture reduces to Conjecture 1.1 (resp. Conjecture 1.3) for $g = 3$ (resp. $g = 5$), and for $g = 4$ strengthens Conjecture 1.2 by allowing odd 5^+-cycles. Moreover, it suggests a lower bound $a(G) \geq \frac{3n}{4}$ if $g \geq 6$, $a(G) \geq \frac{11n}{14}$ if $g \geq 7$, etc. Another way of stating Conjecture 1.4 is to claim that every non-acyclic planar graph G satisfies the inequality

$$\left(1 - \frac{a(G)}{n}\right) g \leq \frac{3}{2}. \tag{1.1}$$

Equivalently, we are looking for the smallest possible constant C, so that

$$\left(1 - \frac{a(G)}{n}\right) g \leq C, \tag{1.2}$$

holds for every planar graph of order n and finite girth g. If true, our conjecture is best possible in the sense that no excluding of a finite set of graphs could yield a better bound.

Indeed, take a tree T and let K be K_4, Q_3 or the dodecahedron. For any graph G obtained by blowing up every vertex of T to a copy of K, (1.1) becomes an equality.

In support to Conjecture 1.4, in the next section we prove that $C = 3$ is sufficient for (1.2).

Theorem 1.5. *If G is a planar graph of order n and girth $g = g(G) < \infty$, then*

$$a(G) > \left(1 - \frac{3}{g}\right) n. \tag{1.3}$$

Moreover, for every integer $g \geq 3$ there exists a planar graph G of girth g for which

$$a(G) = \left\lceil \left(1 - \frac{3}{2g}\right) n \right\rceil. \tag{1.4}$$

Notice that the first part of Theorem 1.5 implies Conjectures 1.1, 1.2, and 1.3, respectively, for girths $g \geq 6$, $g \geq 8$, and $g \geq 10$.

2 Proof of Theorem 1.5

The proof relies on an auxiliary result. Before stating it, let us recall some terminology. We use *k-vertex* and *k^+-vertex* to refer to a vertex of degree k and a vertex of degree at least k, respectively. Given a plane graph $G = (V, E)$, a *face* f is a region of $\mathbb{R}^2 \backslash (V \cup \bigcup E)$, and its *length* $\deg(f)$ is the degree of the corresponding vertex in the geometric dual G^* (thus every bridge incident to f is counted twice in the length); we speak of an *ℓ-face* f if $\deg(f) = \ell$, and an ℓ^+-face is a face of length at least ℓ. Recall that in case of a bridgeless plane graph, every cut-vertex is a 4^+-vertex and for every face f it holds that $\deg(f) = |E(f)|$ (since its topological boundary $\partial(f)$ is a union of simple curves). As usual, we say that a face f is *incident with* a vertex v if $v \in V(f)$. Here is our auxiliary result.

Lemma 2.1. *If G is a simple 2-edge-connected triangle-free plane graph with $\delta(G) \geq 3$, then there exists a face $f \in F(G)$ such that either:*

(i) f is a 4-face incident with at least one 3-vertex; or

(ii) f is a 5-face incident with at least four distinct 3-vertices.

Proof. We use the discharging method. By the Euler formula, it holds that

$$\sum_{v \in V(G)} (\deg(v) - 4) + \sum_{f \in F(G)} (\deg(f) - 4) = -8, \tag{2.1}$$

which leads to the following initial charge $w_0(x)$ for each $x \in V(G) \cup F(G)$:

$$w_0(x) = \deg(x) - 4. \tag{2.2}$$

By (2.1), the total charge is negative. On the other hand, (2.2) tells us that only the 3-vertices are with negative initial charge (equal to -1). Next, redistribute the initial charge according to the following simple rule:

(R) Every 5^+-face sends a charge of $\frac{1}{3}$ to each of its incident 3-vertices.

Let $w_1(x)$ denote the new charge of every $x \in V(G) \cup F(G)$ after applying (R). Assuming that a face satisfying (i) of Lemma 2.1 does not exist, for every $v \in V(G)$ it holds that $w_1(v) \geq 0$ (since G is bridgeless, any 3-vertex lies on the boundary of three faces, thus receives a combined charge of 1). The fact that the total charge remains negative implies the existence of a face f with $w_1(f) < 0$. Moreover, from

$$0 > w_1(f) \geq w_0(f) - \frac{\deg(f)}{3} = \frac{2}{3}(\deg(f) - 6),$$

it follows that every such f must be a 5-face incident with at least four 3-vertices. This completes the proof of the lemma. □

Proof of Theorem 1.5. We show (1.3) by contradiction. Suppose G is a minimal (under inclusion) counter-example to (1.3) among all non-acyclic planar graphs. Then G is clearly connected, of finite girth $g \geq 4$ and $\Delta(G) \geq 3$.

Claim 1: G is bridgeless. For otherwise, let e be a bridge and denote by G_1, G_2 the components of $G - e$. The choice of G combined with the fact that both subgraphs G_1, G_2 are of girth at least g, implies that $a(G_i) > \left(1 - \frac{3}{g}\right) n(G_i)$ for $i = 1, 2$. Summing up leads to the desired contradiction (1.3).

Let $\widetilde{G}$ be a plane embedding of the graph obtained by suppressing every 2-vertex in G. Then $\widetilde{G}$ is bridgeless and $\delta(\widetilde{G}) \geq 3$. Next we show that $\widetilde{G}$ meets all the requirements of Lemma 2.1.

Claim 2: $\widetilde{G}$ is simple and triangle-free. Supposing the opposite, there is a cycle C of G passing through at most three 3^+-vertices. Denote by S the set of 2-vertices in $V(C)$ and set $s = |S|$. In the graph $G' = G - V(C)$, let M be a maximum acyclic set. Then $M \cup S$ is an acyclic set of G, hence $a(G) \geq a(G') + s$. Combined with the choice of G, this would imply that

$$\left(1 - \frac{3}{g}\right)(n - s - 3) + s < \left(1 - \frac{3}{g}\right) n,$$

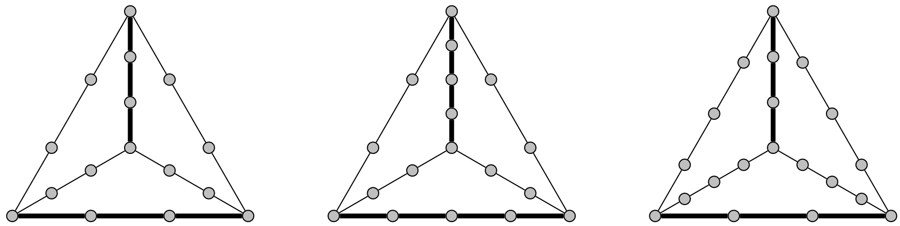
which is equivalent to $s + 3 < g$. However, the last inequality contradicts that the length of C is at least g, and thus settles the claim.

Our aim of contradicting the existence of G is now achievable. Select an $f \in F(\widetilde{G})$ as in Lemma 2.1, and denote $\ell = \deg(f)$. For this choice of f we can certainly find an independent (seen in $\widetilde{G}$) $(\ell - 3)$-subset $T \subseteq V(f)$ consisting entirely of 3-vertices. Indeed, in case $\ell = 4$ the last assertion is trivial; as for $\ell = 5$, it is enough to consider four consecutive 3-vertices v_1, v_2, v_3, v_4 on f and observe that, by planarity, v_1, v_3 or v_2, v_4 form an independent pair.

Returning back to G, every boundary edge of f becomes a path of G whose interior consists entirely of 2-vertices. Let $V_2(f)$ be the collection of all 2-vertices lying on f, and denote $r = |V_2(f)|$. Take from the graph $G' = G - (V(f) \cup V_2(f))$ a maximum acyclic set M. Then $M \cup V_2(F) \cup T$ is an acyclic set of G, giving that $a(G) \geq a(G') + r + \ell - 3$. Similarly to before, the last inequality would imply

$$\left(1 - \frac{3}{g}\right)(n - r - \ell) + r + \ell - 3 < \left(1 - \frac{3}{g}\right) n,$$

which is in turn equivalent to $r + \ell < g$. The last inequality is clearly impossible and thus validates (1.3).



(a) $k = 3, r = 0$ (b) $k = 3, r = 1$ (c) $k = 3, r = 2$

Figure 1: Three cases for G (edges coming from M bolded) when $k = 3$.

In regard to the second assertion of Theorem 1.5, we provide a constructive proof based on the fact that the removal of any two vertices decycles K_4: thus every subdivision of K_4 with order n has acyclic number $a = n - 2$. Given an integer $g \geq 3$, it is of the form $3k + r$ where r equals either 0, 1 or 2. Construct the graph G as follows. Consider a copy of K_4 and select a perfect matching M. If $r = 0$, then subdivide $k - 1$ times every $e \in E(K_4)$; else if $r = 1$, then subdivide k times each $e \in M$ and every other edge $k - 1$ times; finally, if $r = 2$, then subdivide $k - 1$ times each $e \in M$ and every other edge k times (see Fig. 1). In either case the constructed subdivision G has the desired girth g. Moreover, as can be readily checked, its order $n = 6k + 2(r - 1)$ and acyclic number $a = 6k + 2(r - 2)$ satisfy

$$\left(1 - \frac{3}{2g}\right) n = a - 1 + \frac{3}{g}, \tag{2.3}$$

since both sides of (2.3) are equal to $(6k + 2r - 3)(3k + r - 1)/(3k + r)$. Thus, it holds that

$$a = \left\lceil \left(1 - \frac{3}{2g}\right) n \right\rceil. \tag{2.4}$$

Additionally, observe that for $g = 3$, (2.3) becomes equal to a, which confirms that the left-hand side of (1.2) is at least $\frac{3}{2}$. This completes the proof of the theorem. □

3 Concluding remarks and further work

We are fully aware that a technically more involved argument could lower the bound $C \leq 3$ in (1.2), however that was not our main objective.

References

[1] J. Akiyama and M. Watanabe, Maximum induced forests of planar graphs, *Graphs Combin.* **3** (1987), 201–202, doi:10.1007/bf01788541.

[2] M. O. Albertson and D. M. Berman, A conjecture on planar graphs, in: J. A. Bondy and U. S. R. Murty (eds.), *Graph Theory and Related Topics*, Academic Press, New York, p. 357, 1979, proceedings of the conference held in honour of Professor W. T. Tutte on the occasion of his sixtieth birthday.

[3] N. Alon, Problems and results in extremal combinatorics—I, *Discrete Math.* **273** (2003), 31–53, doi:10.1016/s0012-365x(03)00227-9.

[4] N. Alon, D. Mubayi and R. Thomas, Large induced forests in sparse graphs, *J. Graph Theory* **38** (2001), 113–123, doi:10.1002/jgt.1028.

[5] J. A. Bondy and U. S. R. Murty, *Graph Theory*, volume 244 of *Graduate Texts in Mathematics*, Springer, New York, 2008, doi:10.1007/978-1-84628-970-5.

[6] O. V. Borodin, A proof of B. Grünbaum's conjecture on the acyclic 5-colorability of planar graphs, *Dokl. Akad. Nauk SSSR* **231** (1976), 18–20.

[7] R. M. Karp, Reducibility among combinatorial problems, in: R. E. Miller, J. W. Thatcher and J. D. Bohlinger (eds.), *Complexity of Computer Computations*, Plenum Press, New York, The IBM Research Symposia Series, pp. 85–103, 1972, doi:10.1007/978-1-4684-2001-2_9.

[8] Ł. Kowalik, B. Lužar and R. Škrekovski, An improved bound on the largest induced forests for triangle-free planar graphs, *Discrete Math. Theor. Comput. Sci.* **12** (2010), 87–100, https://www.dmtcs.org/dmtcs-ojs/index.php/dmtcs/article/view/1323/0.html.

[9] N. Punnim, The decycling number of regular graphs, *Thai J. Math.* **4** (2006), 145–161, http://thaijmath.in.cmu.ac.th/index.php/thaijmath/article/view/67.

[10] M. R. Salavatipour, Large induced forests in triangle-free planar graphs, *Graphs Combin.* **22** (2006), 113–126, doi:10.1007/s00373-006-0642-7.