Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 14 (2018) 445–454

A note on extremal results on directed acyclic graphs

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Received 17 May 2016, accepted 31 May 2017, published online 6 February 2018

Abstract

This paper studies the maximum number of edges of a Directed Acyclic Graph (DAG) with n vertices in terms of it's longest path ℓ. We prove that in general this number is the Turán number $t(n, l+1)$, the maximum number of edges in a graph with n vertices without a clique of size $\ell+2$. Furthermore, we find the maximum number of edges in a DAG which is either reduced, strongly reduced or extremely reduced and we relate this extremal result with the family of intersection graphs of families of boxes with transverse intersection.

Keywords: Directed graphs, Turán numbers, intersection graphs of families of boxes.

Math. Subj. Class.: 05C20, 52C99

1 Introduction

One of the fundamental results in extremal graph theory is the Theorem of Turán (1941) which states that a graph with n vertices that has more than $t(n, k)$ edges, will always contain a complete subgraph of size $k + 1$. The *Turán graph* $T(n, k)$, is a k–partite graph on n vertices whose partite sets are as nearly equal in cardinality, and has the property

* Partially supported by MTM 2015-63612P.
† Supported by CONACyT 166306.
‡ Partially supported by PAPIIT 104915 and CONACyT 166306.

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that contains the maximum posible number of edges $t(n,k)$ of any graph not containing a clique of size $k+1$. It is known that $t(n,k) \leq (1-\frac{1}{k})\frac{n^2}{2}$, and equality holds if k divides n. In fact, $\lim_{n\to\infty} \frac{t(n,m)}{n^2/2} = 1 - \frac{1}{m}$. See [1].

Turán numbers for several families of graphs have been studied in the context of extremal graph theory, see for example [3] and [4]. In ([2, 7]) the authors analyze, among other things, the intersection graphs of boxes in $\mathbb{R}^d$ proving that, if $\mathcal{T}(n,k,d)$ denotes the maximal number of intersection pairs in a family $\mathcal{F}$ of n boxes in $\mathbb{R}^d$ with the property that no $k+1$ boxes in $\mathcal{F}$ have a point in common (with $n \geq k \geq d \geq 1$), then $\mathcal{T}(n,k,d) = \mathcal{T}(n-k+d,d) + \mathcal{T}(n,k-d+1,1)$, with $\mathcal{T}(n,k,1) = \binom{n}{2} - \binom{n-k+1}{2}$ being the precise bound in dimension 1 for the family of interval graphs.

Turán numbers have played and important role for several variants of the Turán Theorem and its relation with the fractional Helly Theorem (see [5, 6]).

The purpose of this paper is to study the maximum number of edges in directed acyclic graphs with n vertices with respect to it's longest path. That turns out to be related with the extremal behavior of the family of intersection graphs for a collection of boxes in $\mathbb{R}^2$ with transverse intersection.

The first result, Theorem 2.10, states that in a directed acyclic graph with n vertices, if the longest path has length ℓ, then the maximal number of edges is the Turán number $t(n,\ell+1)$.

Theorem 3.19 and its Corollaries state that given a directed acyclic graph $\vec{G}$ with n vertices such that the longest path has length ℓ, then if $\vec{G}$ is either reduced, strongly reduced or extremely reduced, $\vec{G}$ has at most $t(n-\ell+1,2) + \mathcal{T}(n,\ell,1)$ edges, where again $\mathcal{T}(n,\ell,1)$ denotes the maximal number of intersecting pairs in a family $\mathcal{F}$ of n intervals in $\mathbb{R}$ with the property that no $\ell+1$ intervals in $\mathcal{F}$ have a point in common.

In fact, this bound is best possible. The bound is reached by the intersection graph of a collection of boxes in $\mathbb{R}^2$ with transverse intersection (see Proposition 4.6). This graph is extremely reduced (and thus is also strongly reduced and reduced, see Proposition 4.4).

2 Directed acyclic graphs

By a *directed acyclic graph*, DAG, we mean a simple directed graph without directed cycles. A DAG, $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$, with vertex set $\mathcal{V}$ and directed edge set $\vec{\mathcal{E}}$ is *transitive* if for every $x,y,z \in \mathcal{V}$, if $\{x,y\},\{y,z\} \in \vec{\mathcal{E}}$ then $\{x,z\} \in \vec{\mathcal{E}}$.

Definition 2.1. A *topological order* of a directed graph $\vec{G}$ is an ordering $\mathcal{O}$ of its vertices $\{v_1, v_2, \ldots, v_n\}$ so that for every edge $\{v_i, v_j\}$ then $i<j$.

The following proposition is a well known result:

Proposition 2.2. *A directed graph $\vec{G}$ is a DAG if and only if $\vec{G}$ has a topological order.*

Given any set X, by $|X|$ we denote the cardinal of X.

Definition 2.3. The *indegree*, $\deg^-(v)$, of a vertex v is the number of directed edges $\{x,v\}$ with $x \in \mathcal{V}$. The *outdegree*, $\deg^+(v)$, of a vertex v is the number of directed edges $\{v,x\}$ with $x \in \mathcal{V}$. Notice that each directed edge $\{v,w\}$ adds one outdegree to the vertex v and one indegree to the vertex w. Therefore, $\sum_{v\in\mathcal{V}} \deg^+(v) = \sum_{v\in\mathcal{V}} \deg^-(v) = |(\vec{\mathcal{E}})|$.

The degree of a vertex is $\deg(v) = \deg^-(v) + \deg^+(v)$.

Definition 2.4. A vertex v such that $\deg^-(v) = 0$ is called *source*. A vertex v such that $\deg^+(v) = 0$ is called *sink*.

It is well known that every DAG $\vec{G}$ has at least one source and one sink.

Definition 2.5. Given a DAG, $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$, a directed path $\vec{\gamma}$ in G is a sequence of vertices $\{v_0, \dots, v_n\}$ such that $\{v_{i-1}, v_i\} \in \vec{\mathcal{E}}$ for every $1 \le i \le n$. Here, $\vec{\gamma}$ has *length* n, and *endpoint* v_n.

Observe that since DAG's are acyclic, all the vertices on a directed path are different.

Definition 2.6. Given a DAG, $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$, let $\Gamma\colon \mathcal{V} \to \mathbb{N}$ be such that $\Gamma(v) = k$ if there exists a directed path $\vec{\gamma}$ in G of length k with endpoint v and there is no directed path $\vec{\gamma}'$ with endpoint v and length greater than k.

Definition 2.7. Given a DAG, $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$ suppose that $\ell = \max\{k \mid \Gamma(v) = k$ for every $v \in \mathcal{V}\}$. Notice that, since $\vec{G}$ has no directed cycle, $\ell \le |\mathcal{V}|$. Then, let us define a partition $P_\Gamma = \{V_0, \dots, V_\ell\}$ of $\mathcal{V}$ such that $V_i := \{v \in \mathcal{V} \mid \Gamma(v) = i\}$ for every $0 \le i \le \ell$.

Notice that V_0 is exactly the set of sources in $\vec{G}$ and V_ℓ is contained in the set of sinks in G.

Lemma 2.8. *V_i is nonempty for every $0 \le i \le \ell$.*

Proof. Let $\{v_0, \dots, v_\ell\}$ be a directed path of maximal length in $\vec{G}$. Clearly, for every $0 \le i \le \ell$, $v_i \notin V_j$ if $j < i$. Suppose $v_i \in V_j$ with $i < j \le \ell$. Then, there is a directed path $\{v'_0, \dots, v'_j = v_i\}$ with $j > i$ and $\{v'_0, \dots, v'_j, v_{i+1}, \dots, v_\ell\}$ is a directed path with length $j + l - i > \ell$ which contradicts the hypothesis. □

Lemma 2.9. *The induced subgraph with vertices V_i, $G[V_i]$, is independent (has no edges) for every i.*

Proof. Let $v_i, v'_i \in V_i$ and suppose $\{v_i, v'_i\} \in \vec{\mathcal{E}}$. Let $\{v_0, \dots, v_i\}$ be a path of length i with endpoint v_i. Then, $\{v_0, \dots, v_i, v'_i\}$ defines a directed path of length $i+1$ which contradicts the fact that $v'_i \in V_i$. □

Recall that $T(n, \ell)$ denote the ℓ-partite Turán graph with n vertices and $t(n, \ell)$ denote the number of edges of $T(n, \ell)$.

Theorem 2.10. *Let $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$ be a DAG with n vertices such that the longest directed path has length ℓ. Then, $\vec{G}$ has at most $t(n, \ell + 1)$ edges.*

Proof. Consider the partition $P_\Gamma = \{V_0, \dots, V_\ell\}$ of $\mathcal{V}$. By Lemma 2.9, this defines an $(\ell+1)$-partite directed graph. Thus, neglecting the orientation we obtain a complete $(\ell+1)$-partite graph with partition sets $V_0, \dots, V_\ell$. Therefore, the number of edges is at most $t(n, \ell + 1)$. □

Remark 2.11. It is readily seen that the bound in Theorem 2.10 is best possible. Consider the Turán graph $T(n, \ell + 1)$ and any ordering of the $\ell + 1$ independent sets $V_0, \dots, V_\ell$. Then, for every edge $\{v_i, v_j\}$ in $T(n, \ell)$ with $v_i \in V_i$, $v_j \in V_j$ and $i < j$ let us assume the orientation $\{v_i, v_j\}$. It is trivial to check that the resulting graph is a DAG with $t(n, \ell + 1)$ edges.

3 Reduced, strongly reduced and extremely reduced DAGs

Let $\mathcal{O}$ be a topological ordering in a DAG $\vec{G}$. Given any two vertices v, w, and two directed paths in $\vec{G}$, γ,γ', from v to w, let us define $\gamma \cup_{\mathcal{O}} \gamma'$ as the sequence of vertices defined by the vertices in $\gamma \cup \gamma'$ in the order given by $\mathcal{O}$. Of course, this need not be, in general, a directed path from v to w.

Let $\Gamma(u, v)$ be the set of all directed paths from u to v. Let $\cup_{\mathcal{O}}\{\gamma \mid \gamma \in \Gamma(u, v)\}$ represent the sequence of all the vertices from the paths in $\Gamma(u, v)$ ordered according to $\mathcal{O}$.

Definition 3.1. A finite DAG $\vec{G}$ is *strongly reduced* if for any topological ordering $\mathcal{O}$ of $\vec{G}$, every pair of vertices, v, w, and every pair of directed paths, γ, γ', from v to w, then $\gamma \cup_{\mathcal{O}} \gamma'$ defines a directed path from v to w.

Remark 3.2. Let $\vec{G}$ be DAG. Given any two vertices v, w, and two directed paths in $\vec{G}$, γ,γ', from v to w, let us define $\gamma \leq \gamma'$ if every vertex in γ is also in γ'. Clearly, "$\leq$" is a partial order.

Definition 3.3. A vertex w is *reachable* from a vertex v if there is a directed path from v to w.

Proposition 3.4. *Given a finite DAG $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$, the following properties are equivalent:*

i) For every pair of vertices v, w and every pair of paths, γ, γ', from v to w, there exists a directed path from v to w, γ'', such that $\gamma, \gamma' \leq \gamma''$.

ii) For every pair of vertices v, w such that w is reachable from v, there is a directed path from v to w, γ_M, such that for every directed path, γ, from v to w, $\gamma \leq \gamma_M$.

iii) For every topological ordering $\mathcal{O}$ of $\vec{G}$ and any pair of vertices v, w, $\cup_{\mathcal{O}}\{\gamma \mid \gamma \in \Gamma(u, v)\}$ defines a directed path from v to w.

Proof. Since the graph is finite and the relation '$\leq$' is transitive, *i)* and *ii)* are trivially equivalent.

If *ii)* is satisfied, then it is trivial to see that $\cup_{\mathcal{O}}\{\gamma \mid \gamma \in \Gamma(u, v)\} = \gamma_M$ and *iii)* is satisfied. Also, it is readily seen that *iii)* implies *ii)* taking $\gamma_M := \cup_{\mathcal{O}}\{\gamma \mid \gamma \in \Gamma(u, v)\}$. □

Definition 3.5. We say that a finite DAG $\vec{G}$ is *reduced* if it satisfies any of the properties from Proposition 3.4.

Proposition 3.6. *If a finite DAG $\vec{G}$ is strongly reduced, then $\vec{G}$ is reduced.*

Proof. Since the graph is finite, it is immediate to see that being strongly reduced implies *iii)*. □

Remark 3.7. The converse is not true. The graph in the left from Figure 1 is clearly reduced. Notice that the directed path $\gamma_M := \{v_1, v_2, v_3, v_4, v_5\}$ is an upper bound for every directed path from v_1 to v_5. However, if we consider the directed paths $\gamma = \{v_1, v_2, v_5\}$ and $\gamma' = \{v_1, v_4, v_5\}$ with the topological order $\mathcal{O} = \{v_1, v_2, v_3, v_4, v_5\}$, then $\gamma \cup_{\mathcal{O}} \gamma' = \{v_1, v_2, v_4, v_5\}$ which is not a directed path.

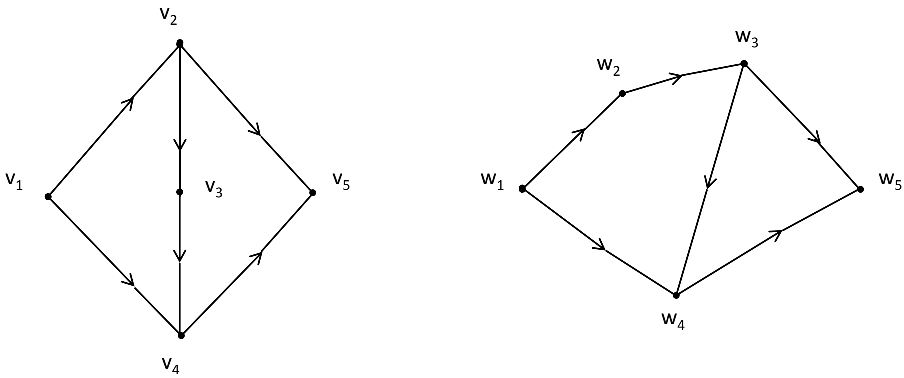

Figure 1: Being reduced does not imply being strongly reduced and being strongly reduced does not imply being extremely reduced.

Definition 3.8. Given a finite DAG $\vec{G}$ and a vertex $v \in \mathcal{V}$ we say that w is an *ancestor* of v if there is a directed path $\{w = v_0, \dots, v_k = v\}$ and w is a *descendant* of v if there is a directed path $\{v = v_0, \dots, v_k = w\}$.

Definition 3.9. We say that a finite DAG $\vec{G}$ is *extremely reduced* if for every pair of non-adjacent vertices x, y, if x, y have a common ancestor, then they do not have a common descendant.

Proposition 3.10. *If a DAG $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$ is extremely reduced, then it is strongly reduced.*

Proof. Let $\gamma = \{v, v_1, \dots, v_n, w\}$ and $\gamma' = \{v, w_0, \dots, w_m, w\}$ be two directed paths in $\vec{G}$ from v yo w. Let $\mathcal{O}$ be any topological order in $\vec{G}$ and consider $\gamma \cup_{\mathcal{O}} \gamma' = \{v, z_1, \dots, z_k, w\}$. First, notice that z_1 is either v_1 or w_1. Therefore, $\{v, z_1\} \in \vec{\mathcal{E}}$. Also, z_k is either v_n or w_m, and $\{z_k, w\} \in \vec{\mathcal{E}}$. Now, for every $1 < i \leq k$, let us see that $\{z_{i-1}, z_i\} \in \vec{\mathcal{E}}$. If $z_{i-1}, z_i \in \gamma$ or $z_{i-1}, z_i \in \gamma'$, then they are consecutive vertices in a directed path and we are done. Otherwise, since z_{i-1}, z_i have a common ancestor v and a common descendant w, then there is a directed edge joining them and, since z_{i-1}, z_i are sorted by a topological order, $\{z_{i-1}, z_i\} \in \vec{\mathcal{E}}$. □

Remark 3.11. The converse is not true. The graph in the right from Figure 1 is strongly reduced. However, vertices w_2 and w_4 are not adjacent and have a common ancestor and a common descendent.

Proposition 3.12. *If $\vec{G}$ is transitive, then the following properties are equivalent:*

- *$\vec{G}$ is extremely reduced,*
- *$\vec{G}$ is strongly reduced,*
- *$\vec{G}$ is reduced.*

Proof. By Proposition 3.10 if $\vec{G}$ is extremely reduced, then it is strongly reduced. By Proposition 3.6, if $\vec{G}$ is strongly reduced, then it is reduced.

Suppose $\vec{G}$ is reduced and suppose that two vertices x, y have a common ancestor, v, and a common descendant, w. Then, there are two directed paths γ, γ' from v to w such that $x \in \gamma$ and $y \in \gamma'$. By property *i)* in Proposition 3.4, there exists a path γ'' in $\vec{G}$ from v to w such that $\gamma, \gamma' \leq \gamma''$. In particular, $x, y \in \gamma''$. Therefore, either x is reachable from y or y is reachable from x in $\vec{G}$. Since $\vec{G}$ is transitive, this implies that x, y are adjacent. Therefore, $\vec{G}$ is extremely reduced. □

Definition 3.13. Given a DAG $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$, the graph with vertex set $\mathcal{V}$ and edge set $\vec{\mathcal{E}}' := \vec{\mathcal{E}} \cup \{\{v, w\} \mid w \text{ is reachable from } v\}$ is called the *transitive closure* of $\vec{G}$, $T[\vec{G}]$.

It is immediate to check the following:

Proposition 3.14. *Given any DAG $\vec{G}$, $T[\vec{G}]$ is transitive.*

Proposition 3.15. *If a DAG $\vec{G}$ is reduced, then the transitive closure $T[\vec{G}]$ is also reduced.*

Proof. Suppose $\vec{G}$ satisfies *i)* in Proposition 3.4 and let $\gamma = \{v = v_0, \ldots, v_n = w\}$, $\gamma' = \{v = w_0, \ldots, w_m = w\}$ be any pair of paths from v to w in $T[\vec{G}]$. Therefore, v_i is reachable from v_{i-1} in $\vec{G}$ for every $1 \leq i \leq n$ and w_i is reachable from w_{i-1} in $\vec{G}$ for every $1 \leq i \leq m$. Thus, there exist a path γ_0 in $\vec{G}$ such that $\gamma \leq \gamma_0$ and a path γ_0' in $\vec{G}$ such that $\gamma' \leq \gamma_0'$. By property *i)*, there is a directed path from v to w such that $\gamma_0, \gamma_0' \leq \gamma_0''$. Therefore, $\gamma, \gamma' \leq \gamma_0''$ and $T[\vec{G}]$ satisfies *i)*. □

Then, from Propositions 3.6, 3.10, 3.12, 3.14 and 3.15:

Corollary 3.16. *If a DAG $\vec{G}$ is reduced, then the transitive closure $T[\vec{G}]$ is extremely reduced and strongly reduced. In particular, if $\vec{G}$ is extremely reduced or strongly reduced, then $T[\vec{G}]$ is extremely reduced and strongly reduced.*

Let us recall that

$$\mathcal{T}(n, \ell, 1) = \binom{n}{2} - \binom{n - \ell + 1}{2} = (n - \ell + 1)(\ell - 1) + \frac{(\ell - 1)(\ell - 2)}{2} \tag{3.1}$$

As it was proved in [7]:

Lemma 3.17. *For $n \geq \ell$ and $d \geq 1$,*

$$\mathcal{T}(n + d, \ell, 1) - \mathcal{T}(n, \ell, 1) = d(\ell - 1).$$

In particular, $\mathcal{T}(n + 2, \ell, 1) - \mathcal{T}(n, \ell, 1) = 2(\ell - 1)$.
Also, from [7]:

Lemma 3.18. *For $1 \leq d \leq n$,*

$$t(n + d, d) - t(n, d) = (d - 1)n + \binom{d}{2}$$

In particular, $t(n + 2, 2) - t(n, 2) = n + 1$.

Theorem 3.19. *Let $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$ be a DAG with n vertices and such that the longest directed path has length $\ell \geq 1$. If $\vec{G}$ is extremely reduced, then $\vec{G}$ has at most $t(n - \ell + 1, 2) + \mathcal{T}(n, \ell, 1)$ edges.*

Proof. Let us prove the result by induction on n. Suppose that the longest directed path has length ℓ.

First, let us see that the result is true for $n = \ell + 1$ and $n = \ell + 2$.

If $n = \ell + 1$ then $\vec{G}$ has at most $\frac{\ell(\ell+1)}{2} = \frac{(\ell-2)(\ell-1)}{2} + 2(\ell - 1) + 1 = \mathcal{T}(n, \ell, 1) + t(n - \ell + 1, 2)$ edges. The last equation follows immediately from (3.1) and the fact that $t(2, 2) = 1$.

If $n = \ell + 2$ then there are $\ell + 1$ vertices which define a directed path $\gamma = \{v_0, \ldots, v_\ell\}$ and one vertex w such that neither $\{w, v_0\}$ nor $\{v_\ell, w\}$ is a directed edge. Then, the partition $P_\Gamma = \{V_0, \ldots, V_\ell\}$ of $\vec{G}$ satisfies that $v_i \in V_i$ for every $0 \leq i \leq \ell$. Also, $w \in V_j$ for some $0 \leq j \leq \ell$ and $\{w, v_j\}$, $\{v_j, w\}$ are not directed edges. Hence, $\deg(w) \leq \ell$. Therefore, $\vec{G}$ has at most $\frac{\ell(\ell+1)}{2} + \ell = \frac{(\ell-2)(\ell-1)}{2} + 3(\ell - 1) + 2 = \mathcal{T}(n, \ell, 1) + t(n - \ell + 1, 2)$ edges. The last equation follows immediately from (3.1) and the fact that $t(3, 2) = 2$.

Suppose the induction hypothesis holds when the graph has n vertices and let $\#(\mathcal{V}) = n + 2$. Also, by Proposition 3.15 we may assume that the graph is transitive.

Consider the partition $P_\Gamma = \{V_0, \ldots, V_\ell\}$ of $\mathcal{V}$. Let $\#(V_i) = r_i$. Let $v \in V_0$ and w be any sink of $\vec{G}$. Consider any pair of vertices $v_i, v_i' \in V_i$. Since $\vec{G}$ is extremely reduced and every two vertices in V_i are non-adjacent, v_i, v_i' can not be both descendants of v and ancestors of w simultaneously. Hence, the number of edges joining the sets $\{v, w\}$ and V_i are at most $r_i + 1$. Therefore, there are at most $n + \ell - 1$ edges joining $\{v, w\}$ and $G \setminus \{v, w\}$

Since $G \setminus \{v, w\}$ has n vertices, by hypothesis, it contains at most $t(n - \ell + 1, 2) + \mathcal{T}(n, \ell, 1)$ edges.

Finally, there is at most 1 edge in the subgraph induced by $\{v, w\}$.

Therefore, by Lemmas 3.17 and 3.18, $|\vec{E}(G)| \leq t(n-\ell+1, 2) + \mathcal{T}(n, \ell, 1) + n + \ell = t(n - \ell + 3, 2) + \mathcal{T}(n + 2, \ell, 1)$. □

By Corollary 3.16 we know that the extremal graph for reduced and strongly reduced graphs is transitive. Thus, from Theorem 3.19 and Proposition 3.12 we obtain the following.

Corollary 3.20. *Let $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$ be a DAG with n vertices and such that the longest directed path has length $\ell \geq 1$. If $\vec{G}$ is reduced, then $\vec{G}$ has at most $t(n - \ell + 1, 2) + \mathcal{T}(n, \ell, 1)$ edges.*

Corollary 3.21. *Let $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$ be a DAG with n vertices and such that the longest directed path has length $\ell \geq 1$. If $\vec{G}$ is strongly reduced, then $\vec{G}$ has at most $t(n - \ell + 1, 2) + \mathcal{T}(n, \ell, 1)$ edges.*

4 Directed intersection graphs of boxes

Definition 4.1. Let $\mathcal{R}$ be a collection of boxes with parallel axes in $\mathbb{R}^2$. Let $\vec{G} = (\mathcal{V}, \vec{\mathcal{E}})$ be a directed graph such that $\mathcal{V} = \mathcal{R}$ and given $R, R' \in \mathcal{R}$ with $R = I \times J$, $R' = I' \times J'$ then $\{R, R'\} \in \vec{\mathcal{E}}$ if and only if $I \subset I'$ and $J' \subset J$ (i.e. there is an edge if and only if the intersection is *transverse* and the order is defined by the subset relation in the first coordinate). Let us call $\vec{G}$ the *directed intersection graph* of $\mathcal{R}$.

Definition 4.2. Let $\mathcal{R}$ be a collection of boxes with parallel axes in $\mathbb{R}^2$. We say that $\mathcal{R}$ is a collection with *transverse intersection* if for every pair of boxes either they are disjoint or their intersection is transverse.

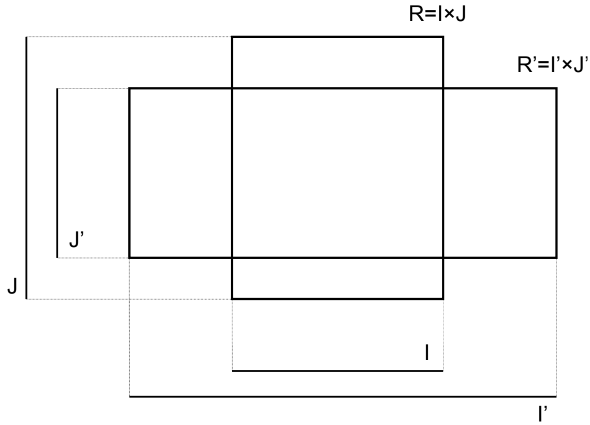

Figure 2: The transverse intersection above induces a directed edge $\{R, R'\}$.

Proposition 4.3. *Let $\mathcal{R}$ be a collection of boxes with parallel axes in $\mathbb{R}^2$ and $\vec{G}$ be the induced directed intersection graph. If two vertices v, w have both a common ancestor and a common descendant in $\vec{G}$, then the corresponding boxes R_v, R_w intersect.*

Proof. Let a be a common ancestor and $R_a = I_a \times J_a$ be the corresponding box. Let b be a common descendant and $R_b = I_b \times J_b$ be the corresponding box. Then if $R_v = I_v \times J_v$, $R_w = I_w \times J_w$ are the boxes corresponding to v and w respectively, it follows by construction that $I_a \subset I_v, I_w$ and $J_b \subset J_v, J_w$. Therefore, $I_a \times J_b \subset R_v, R_w$ and $R_v \cap R_w \neq \emptyset$. □

Proposition 4.4. *If $\mathcal{R}$ is a collection of boxes with parallel axes in $\mathbb{R}^2$ with transverse intersection, then the induced directed intersection graph G is extremely reduced and transitive.*

Proof. First notice that the transitivity holds simply by the transverse intersection property. Let v, w be two vertices such that there is no edge joining them. This means, by construction, that their corresponding boxes do not have a transverse intersection. Since $\mathcal{R}$ has transverse intersection, this implies that these boxes do not intersect. Thus, by Proposition 4.3, if v, w have a common ancestor, then they can not have a common descendant. □

Remark 4.5. Consider the bipartite graph G from Figure 3 with the partition given by $\{\text{letters}, \text{numbers}\}$ and assume all directed edges go from letters into numbers. Note that G is extremely reduced, transitive and acyclic. Notice that the induced subgraphs given by the sets $C_1 := \{1, 2, A, B\}$, $C_2 := \{3, 4, C, D\}$ and $C_3 := \{5, 6, E, F\}$ are three cycles of length 4. Furthermore the induced subgraph given by the set of vertices $\{1, 2, 3, 4, 8, 9, A, B, C, D, H, I\}$ is realizable as boxes in $\mathbb{R}^2$ (see Figure 4) note, that contains C_1 and C_2 and its realization force one of them to be inside the other say C_1 inside C_2. Similarly the induced subgraphs given by the set of vertices $\{1, 2, 5, 6, A, B, E, F, 7, 12, G, L\}$ and the set of vertices $\{3, 4, 5, 6, C, D, E, F, 10, 11, J, K\}$ forces necessarily a system of tree squares one inside the other. However, intervals given by $\{7, 8, 9, 10, 11, 12\}$ and $\{G, H, I, J, K, L\}$ are forced to have more intersections that those given by the graph. In

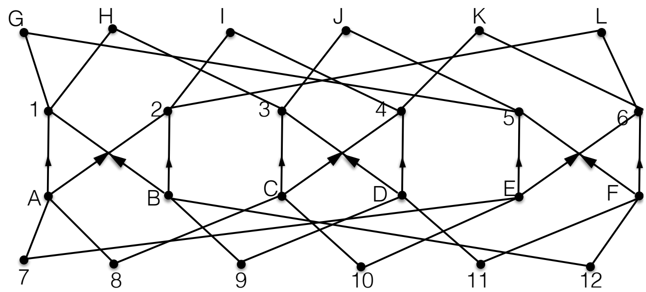

Figure 3: The bipartite, transitive, and extremely reduced DAG, G with partition given by $\{\text{letters}, \text{numbers}\}$ and edges directed from letters into numbers. This graph is not realizable as a family of boxes in $\mathbb{R}^2$.

other words, there is no family of boxes (or intervals) that realizes such a graph or for which it is induced the graph G. Then, the converse of Proposition 4.4 is not true.

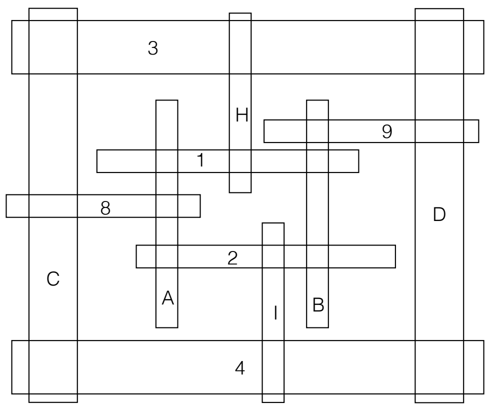

Figure 4: Realization in $\mathbb{R}^2$ of the induced subgraph with vertices $\{1, 2, 3, 4, 8, 9, A, B, C, D, H, I\}$ of the graph shown in Figure 3.

Let $G[r, l, s]$ be the graph, $G(\mathcal{V}, \vec{\mathcal{E}})$, such that:

- $\mathcal{V} = \{x_1, \ldots, x_r, y_1, \ldots, y_{l-1}, z_1, \ldots, z_s\}$
- $\{x_i, x_j\} \notin \vec{\mathcal{E}}$ for any $i \neq j$,
- $\{z_i, z_j\} \notin \vec{\mathcal{E}}$ for any $i \neq j$,
- $\{x_i, y_j\} \in \vec{\mathcal{E}}$ for every i, j,
- $\{y_i, y_j\} \in \vec{\mathcal{E}}$ for every $i < j$,
- $\{y_i, z_j\} \in \vec{\mathcal{E}}$ for every i, j,
- $\{x_i, z_j\} \in \vec{\mathcal{E}}$ for every i, j.

This is the directed intersection graph from the collection of boxes in Figure 5.

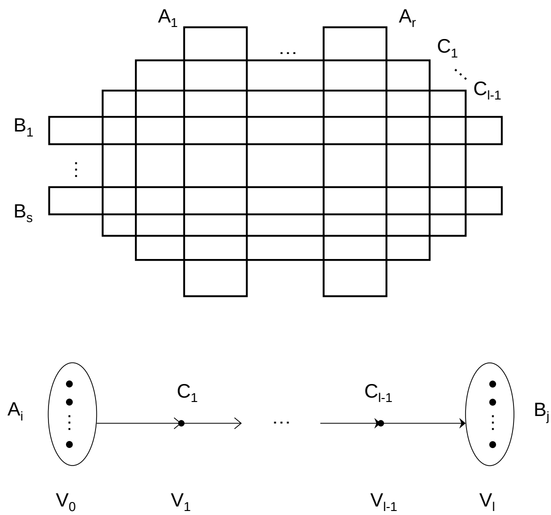

Figure 5: The graph $G[r, l, s]$ corresponds to the directed intersection graph of the collection in the figure where $x_i \sim A_i$, $y_j \sim C_j$ and $z_k \sim B_k$. Notice that the graph is transitive although not every edge is represented in the figure.

By Proposition 4.4, $G[r, l, s]$ is a transitive extremely reduced DAG. In particular, $G[r, l, s]$ is strongly reduced and reduced.

Now, to prove that the bound obtained in Theorem 3.19 and its corollaries is best possible, it is immediate to check the following:

Proposition 4.6. *If $n - \ell$ is even, $G[\frac{n-\ell}{2}, \ell, \frac{n-\ell}{2}]$ has $t(n - \ell + 1, 2) + \mathcal{T}(n, \ell, 1)$ edges. If $n - \ell$ is odd, $G[\frac{n-\ell+1}{2}, \ell, \frac{n-\ell-1}{2}]$ has $t(n - \ell + 1, 2) + \mathcal{T}(n, \ell, 1)$ edges.*

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