Octahedral, dicyclic and special linear solutions of some Hamilton-Waterloo problems

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Abstract

We give a sharply-vertex-transitive solution of each of the nine Hamilton-Waterloo problems left open by Danziger, Quattrocchi and Stevens.

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1 Introduction

A cycle decomposition of a simple graph $\Gamma = (V, E)$ is a set $D$ of cycles whose edges partition $E$. A partition $\mathcal{F}$ of $D$ into classes (2-factors) each of which covers all $V$ exactly once is said to be a 2-factorization of $\Gamma$. The type of a 2-factor $F$ is the partition $\pi = [\ell_1^{n_1}, \ldots, \ell_t^{n_t}]$ (written in exponential notation) of the integer $|V|$ into the lengths of the cycles of $\mathcal{F}$.

A 2-factorization $\mathcal{F}$ of $K_v$ (the complete graph of order $v$) or $K_v - I$ (the cocktail party graph of order $v$) whose 2-factors are all of the same type $\pi$ is a solution of the so-called Oberwolfach Problem OP$(v; \pi)$. If instead the 2-factors of $\mathcal{F}$ are of two different types $\pi$ and $\psi$, then $\mathcal{F}$ is a solution of the so-called Hamilton-Waterloo Problem HWP$(v; \pi, \psi; r, s)$ where $r$ and $s$ denote the number of 2-factors of $\mathcal{F}$ of type $\pi$ and $\psi$, respectively.

A complete solution of the OPs whose 2-factors are uniform, namely of the form OP$(\ell n; [\ell^n])$, has been given in [1] and [12]. Other important classes of OPs has been...
solved in [4, 15]. For the time being, to look for a solution to all possible OPs and, above all, HWPs is too ambitious. Anyway it is reasonable to believe that we are not so far from a complete solution of the HWPs whose 2-factors are uniform, namely of the form HWP\((v; [h^v/h], [w^v/w]; r, s)\). We can say this especially because of the big progress recently done in [10].

Danziger, Quattrocchi and Stevens [11] treated the HWPs whose 2-factors are either triangle-factors or quadrangle-factors, they namely studied HWP\((12n; [3^{4n}], [4^{3n}]; r, s)\). In the following such an HWP will be denoted, more simply, by HWP\((12n; 3, 4; r, s)\). They solved this problem for all possible triples \((n, r, s)\) except the following ones:

(i) \((4, r, 23 − r)\) with \(r \in \{5, 7, 9, 13, 15, 17\}\);

(ii) \((2, r, 11 − r)\) with \(r \in \{5, 7, 9\}\).

Six of the nine above problems have been recently solved in [14] where it was pointed out that all nine problems were also solved in a work still in preparation [2] by the authors of the present paper. Meanwhile, a solution for each of the remaining three problems not considered in [14] have been given in [16]. Notwithstanding, in the present paper we want to present our solutions to the nine HWPs left open by Danziger, Quattrocchi and Stevens in detail. These solutions, differently from those of [14, 16], are full of symmetries since they are \(G\)-regular for a suitable group \(G\). We recall that a cycle decomposition (or 2-factorization) of a graph \(\Gamma\) is said to be \(G\)-regular when it admits \(G\) as an automorphism group acting sharply transitively on all vertices. Here is explicitly our main result:

**Theorem 1.1.** There exists a \(\overline{O}\)-regular 2-factorization of \(K_{48} − I\) having \(r\) triangle-factors and \(23 − r\) quadrangle-factors where \(\overline{O}\) is the binary octahedral group and \(r \in \{5, 7, 9, 13, 15, 17\}\).

There exists a \(Q_{24}\)-regular 2-factorization of \(K_{24} − I\) having \(r\) triangle-factors and \(11 − r\) quadrangle-factors where \(Q_{24}\) is the dicyclic group of order \(24\) and \(r \in \{7, 9\}\).

There exists a \(SL_2(3)\)-regular 2-factorization of \(K_{24} − I\) having six triangle-factors and five quadrangle-factors where \(SL_2(3)\) is the 2-dimensional special linear group over \(\mathbb{Z}_3\).

## 2 Some preliminaries

The use of the classic method of differences allowed to get cyclic (namely \(Z_v\)-regular) solutions of some HWPs in [8, 9, 13]. Now we summarize, in the shortest possible way, the method of partial differences. This method, explained in [7] and successfully applied in many papers (see, especially, [6]), has been also useful for the investigation of \(G\)-regular 2-factorizations of a complete graph of odd order [9]. The \(G\)-regular 2-factorizations of a cocktail party graph can be treated similarly.

Throughout this paper any group \(G\) will be assumed to be written multiplicatively and its identity element will be denoted by \(1\). Let \(\Omega\) be a symmetric subset of a group \(G\); this means that \(1 \notin \Omega\) and that \(\omega \in \Omega\) if and only if \(\omega^{-1} \in \Omega\). The Cayley graph on \(G\) with connection-set \(\Omega\), denoted by \(\text{Cay}[G : \Omega]\), is the simple graph whose vertices are the elements of \(G\) and whose edges are all 2-subsets of \(G\) of the form \(\{g, \omega g\}\) with \((g, \omega) \in G \times \Omega\).

**Remark 2.1.** If \(\lambda\) is an involution of a group \(G\), then \(\text{Cay}[G : G \setminus \{1, \lambda\}])\) is isomorphic to \(K_{|G|} − I\). So, in the following, such a Cayley graph will be always identified with the cocktail party graph of order \(|G|\).
Let \( \text{Cycle}(G) \) be the set of all cycles with vertices in \( G \) and consider the natural right action of \( G \) on \( \text{Cycle}(G) \) defined by \((c_1, c_2, \ldots, c_n)^g = (c_1g, c_2g, \ldots, c_ng)\) for every \( C = (c_1, c_2, \ldots, c_n) \in \text{Cycle}(G) \) and every \( g \in G \). The stabilizer and the orbit of any \( C \in \text{Cycle}(G) \) under this action will be denoted by \( \text{Stab}(C) \) and \( \text{Orb}(C) \), respectively. The list of differences of \( C \in \text{Cycle}(G) \) is the multiset \( \Delta C \) of all possible quotients \( xy^{-1} \) with \((x,y)\) an ordered pair of adjacent vertices of \( C \). One can see that the multiplicity \( m_{\Delta C}(g) \) of any element \( g \in G \) in \( \Delta C \) is a multiple of the order of \( \text{Stab}(C) \). Thus it makes sense to speak of the list of partial differences of \( C \) as the multiset \( \partial C \) on \( G \) in which the multiplicity of any \( g \in G \) is defined by

\[
m_{\partial C}(g) := \frac{m_{\Delta C}(g)}{|\text{Stab}(C)|}.
\]

We underline the fact that \( \partial C \) is, in general, a multiset. Note that if \( \partial C \) is a set, namely without repeated elements, then it is symmetric so that it makes sense to speak of the Cayley graph \( \text{Cay}[G : \partial C] \). The following elementary but crucial result holds.

**Lemma 2.2.** If \( C \in \text{Cycle}(G) \) and \( \partial C \) does not have repeated elements, then \( \text{Orb}(C) \) is a \( G \)-regular cycle-decomposition of \( \text{Cay}[G : \partial C] \).

By Remark 2.1, as an immediate consequence of the above lemma we can state the following result.

**Theorem 2.3.** Let \( \lambda \) be an involution of a group \( G \). If \( \{C_1, \ldots, C_t\} \) is a subset of \( \text{Cycle}(G) \) such that \( \bigcup_{i=1}^t \partial C_i = G \setminus \{1, \lambda\} \), then \( \bigcup_{i=1}^t \text{Orb}(C_i) \) is a \( G \)-regular cycle-decomposition of \( K_{|G|} - I \).

We need, as last ingredient, the following easy remarks.

**Remark 2.4.** If \( C \in \text{Cycle}(G) \) and \( V(C) \) is a subgroup of \( G \), then \( \text{Orb}(C) \) is a 2-factor of the complete graph on \( G \) whose stabilizer is the whole \( G \).

If \( C_1, \ldots, C_t \) are cycles of \( \text{Cycle}(G) \) and \( \bigcup_{i=1}^t V(C_i) \) is a complete system of representatives for the left cosets of a subgroup \( S \) of \( G \), then \( \bigcup_{i=1}^t \text{Orb}_S(C_i) \) is a 2-factor of the complete graph on \( G \) whose stabilizer is \( S \).

## 3 Octahedral solutions of six Hamilton-Waterloo problems

Throughout this section \( G \) will denote the so-called binary octahedral group which is usually denoted by \( \bar{O} \). This group, up to isomorphism, can be viewed as a group of units of the skew-field \( \mathbb{H} \) of quaternions introduced by Hamilton, that is an extension of the complex field \( \mathbb{C} \). We recall the basic facts regarding \( \mathbb{H} \). Its elements are all real linear combinations of \( 1, i, j \) and \( k \). The sum and the product of two quaternions are defined in the natural way under the rules that

\[
i^2 = j^2 = k^2 = ijk = -1.
\]

If \( q = a + bi + cj + dk \neq 0 \), then the inverse of \( q \) is given by

\[
q^{-1} = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}.
\]
The 48 elements of the multiplicative group $G$ are the following:

$$\pm 1, \pm i, \pm j, \pm k;$$
$$\frac{1}{2}(\pm 1 \pm i \pm j \pm k);$$
$$\frac{1}{\sqrt{2}}(\pm x \pm y), \{x, y\} \in \left(\binom{1, i, j, k}{2}\right).$$

The use of the octahedral group $G$ was crucial in [3] to get a Steiner triple system of any order $v = 96n + 49$ with an automorphism group acting sharply transitively an all but one point. Here $G$ will be used to get a $G$-regular solution of each of the six Hamilton-Waterloo problems of order 48 left open in [11]. We will need to consider the following subgroups of $G$ of order 16 and 12, respectively:

- $K = \langle k, \frac{1}{\sqrt{2}}(j - k) \rangle$;
- $L = \langle \frac{1}{\sqrt{2}}(j - k), \frac{1}{2}(-1 - i + j + k) \rangle$.

### 3.1 An octahedral solution of HWP(48; 3; 4; 5, 18)

Consider the nine cycles of $Cycle(G)$ defined as follows.

- $C_1 = (1, -\frac{1}{\sqrt{2}}(1 - k), \frac{1}{2}(1 - i - j - k))$
- $C_2 = (1, \frac{1}{2}(-1 - i + j + k), \frac{1}{2}(-1 + i - j - k))$
- $C_3 = (1, \frac{1}{2}(-1 + i + j - k), \frac{1}{2}(-1 - i - j + k))$
- $C_4 = (1, k, -1, -k)$
- $C_5 = (1, j, -1, -j)$
- $C_6 = (1, \frac{1}{\sqrt{2}}(-i + k), -\frac{1}{2}(1 + i + j + k), -\frac{1}{\sqrt{2}}(j + k))$
- $C_7 = (1, \frac{1}{\sqrt{2}}(i - j), \frac{1}{\sqrt{2}}(1 + i), \frac{1}{2}(1 - i - j + k))$
- $C_8 = (1, \frac{1}{2}(1 - i + j - k), k, -\frac{1}{\sqrt{2}}(1 + j))$
- $C_9 = (1, \frac{1}{\sqrt{2}}(1 - i), -\frac{1}{2}(1 + i), \frac{1}{2}(-1 - i + j - k))$

We note that $Stab(C_i) = V(C_i)$ for $2 \leq i \leq 5$ while all other $C_i$’s have trivial stabilizer. Thus, by Lemma 2.2, one can check that $Orb(C_i)$ is a $\ell_i$-cycle decomposition of $Cay[G : \Omega_i]$ where $\ell_i$ is the length of $C_i$ and where the $\Omega_i$’s are the symmetric subsets of $G$ listed below.

- $\Omega_1 = \{-\frac{1}{\sqrt{2}}(1 - k), \frac{1}{2}(1 - i - j - k), -\frac{1}{\sqrt{2}}(1 + i)\}^{\pm 1}$
- $\Omega_2 = \{\frac{1}{2}(-1 - i + j + k)\}^{\pm 1}$
- $\Omega_3 = \{\frac{1}{2}(-1 + i + j - k)\}^{\pm 1}$
- $\Omega_4 = \{k\}^{\pm 1}$
- $\Omega_5 = \{j\}^{\pm 1}$
- $\Omega_6 = \{\frac{1}{\sqrt{2}}(-i + k), \frac{1}{\sqrt{2}}(j - k), \frac{1}{\sqrt{2}}(1 - k), -\frac{1}{\sqrt{2}}(j + k)\}^{\pm 1}$
Lemma 2.2, one can check that Orb length 3 or 1 or 4 according to whether
By Remark 2.4, each cycles of F C G

One can see that the Ωi’s partition G \ {1, −1}. Thus, by Theorem 2.3 we can say that C := \bigcup_{i=1}^{9} OrbG(C_i) is a G-regular cycle-decomposition of K_{48} − I. Now set F_i = Orb_{S_i}(C_i) where

S_i = \begin{cases} K & \text{for } i = 1; \\ G & \text{for } 2 \leq i \leq 5; \\ L & \text{for } 6 \leq i \leq 9. \end{cases}

By Remark 2.4, each F_i is a 2-factor of K_{48} − I with Stab(F_i) = S_i, hence Orb(F_i) has length 3 or 1 or 4 according to whether i = 1, or 2 \leq i \leq 5, or 6 \leq i \leq 9, respectively. The cycles of F_i are triangles or quadrangles according to whether or not i \leq 3. Thus, recalling that C is a cycle-decomposition of K_{48} − I, we conclude that F := \bigcup_{i=1}^{9} Orb(F_i) is a G-regular 2-factorization of K_{48} − I with 5 triangle-factors and 18 quadrangle-factors, namely a G-regular solution of HWP(48; 3, 4; 7, 16).

3.2 An octahedral solution of HWP(48; 3, 4; 7, 16)

Consider the seven cycles of Cycle(G) defined as follows.

C_1 = (1, −\frac{1}{\sqrt{2}}(i + j), \frac{1}{2}(1 + i + j + k))
C_2 = (1, \frac{1}{2}(-1 + i + j + k), \frac{1}{2}(1 - i - j - k))
C_3 = (1, \frac{1}{2}(-1 + i + j + k), \frac{1}{2}(-1 - i - j + k))
C_4 = (1, \frac{1}{\sqrt{2}}(-i + k), \frac{1}{2}(1 + i + j - k), -\frac{1}{\sqrt{2}}(j + k))
C_5 = (1, \frac{1}{\sqrt{2}}(i - j), \frac{1}{\sqrt{2}}(1 - k), \frac{1}{\sqrt{2}}(1 + i))
C_6 = (1, \frac{1}{\sqrt{2}}(1 + k), -\frac{1}{2}(1 + i + j + k), \frac{1}{\sqrt{2}}(1 + j))
C_7 = (1, −\frac{1}{2}(1 + i + j + k), \frac{1}{2}(1 - i + j - k), \frac{1}{2}(1 - i - j + k))

We note that Stab(C_3) = V(C_3) while all other C_i’s have trivial stabilizer. Thus, by Lemma 2.2, one can check that Orb(C_i) is a \ell_i-cycle decomposition of Cay[G : \Omega_i] where \ell_i is the length of C_i and where the \Omega_i’s are the symmetric subsets of G listed below.

\Omega_1 = \{-\frac{1}{\sqrt{2}}(i + j), \frac{1}{2}(1 - i + j + k), \frac{1}{\sqrt{2}}(j + k)\}^{±1}
\Omega_2 = \{\frac{1}{2}(-1 - i + j + k), \frac{1}{2}(1 - i - j - k), \frac{1}{2}(1 - i + j - k)\}^{±1}
\Omega_3 = \{\frac{1}{2}(-1 + i + j - k)\}^{±1}
\Omega_4 = \{\frac{1}{\sqrt{2}}(-i + k), -\frac{1}{\sqrt{2}}(1 - k), \frac{1}{\sqrt{2}}(i + k), -\frac{1}{\sqrt{2}}(j + k)\}^{±1}
\Omega_5 = \{\frac{1}{\sqrt{2}}(i - j), -j, \frac{1}{2}(1 - i + j - k), \frac{1}{\sqrt{2}}(1 + i)\}^{±1}
\Omega_6 = \{\frac{1}{\sqrt{2}}(1 + k), \frac{1}{\sqrt{2}}(-1 + j), -\frac{1}{\sqrt{2}}(1 + i), \frac{1}{\sqrt{2}}(1 + j)\}^{±1}
\Omega_7 = \{-\frac{1}{2}(1 + i + j + k), -i, -k, \frac{1}{2}(1 - i - j + k)\}^{±1}
By Remark 2.4, each $F_i$ is a 2-factor of $K_{48} - I$ with $Stab_G(F_i) = G_i$, hence $Orb_G(F_i)$ has length 3 or 1 or 4 according to whether $i = 1, 2$ or $i = 3$ or $4 \leq i \leq 7$, respectively.

The cycles of $F_i$ are triangles or quadrangles according to whether or not $i \leq 3$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{7} Orb_G(F_i)$ is a $G$-regular 2-factorization of $K_{48} - I$ with 7 triangle-factors and 16 quadrangle-factors, namely a $G$-regular solution of HWP($48; 3, 4, 7, 16$).

3.3 An octahedral solution of HWP($48; 3, 4, 9, 14$)

Consider the eight cycles of $Cycle(G)$ defined as follows.

$$C_1 = (1, \sqrt{2}(i + j), \frac{1}{2}(1 - i - j - k))$$
$$C_2 = (1, -\sqrt{2}(1 - k), \frac{1}{\sqrt{2}}(1 + j))$$
$$C_3 = (1, \frac{1}{2}(-1 - i + j + k), \frac{1}{2}(1 + i - j + k))$$
$$C_4 = (1, \frac{1}{\sqrt{2}}(-i + k), \frac{1}{\sqrt{2}}(1 - i), \frac{1}{2}(-1 - i + j - k))$$
$$C_5 = (1, \frac{1}{\sqrt{2}}(i - j), \frac{1}{2}(-1 + i + j + k), -\frac{1}{\sqrt{2}}(j + k))$$
$$C_6 = (1, \frac{1}{\sqrt{2}}(1 + i), \frac{1}{\sqrt{2}}(1 - i), \frac{1}{2}(1 - i - j + k))$$
$$C_7 = (1, k, -1, -k)$$
$$C_8 = (1, j, -1, -j)$$

We note that $Stab(C_i) = V(C_i)$ for $i = 7, 8$ while all other $C_i$’s have trivial stabilizer. By Lemma 2.2, one can check that $Orb(C_i)$ is a $\ell_i$-cycle decomposition of Cay($G : \Omega_i$) where $\ell_i$ is the length of $C_i$ and where the $\Omega_i$’s are the symmetric subsets of $G$ listed below.

$$\Omega_1 = \{ \frac{1}{\sqrt{2}}(i + j), \frac{1}{2}(1 - i - j - k), \frac{1}{\sqrt{2}}(-1 + i) \}^{\pm 1}$$
$$\Omega_2 = \{ -\frac{1}{\sqrt{2}}(1 - k), \frac{1}{\sqrt{2}}(1 + j), \frac{1}{2}(-1 + i + j + k) \}^{\pm 1}$$
$$\Omega_3 = \{ \frac{1}{2}(-1 - i + j + k), \frac{1}{2}(1 + i - j + k), \frac{1}{2}(-1 - i - j + k) \}^{\pm 1}$$
$$\Omega_4 = \{ \frac{1}{\sqrt{2}}(-i + k), \frac{1}{2}(1 - i + j + k), \frac{1}{\sqrt{2}}(i + k), \frac{1}{2}(-1 - i + j - k) \}^{\pm 1}$$
$$\Omega_5 = \{ \frac{1}{\sqrt{2}}(i - j), \frac{1}{\sqrt{2}}(j - k), -\frac{1}{\sqrt{2}}(1 + j), -\frac{1}{\sqrt{2}}(j + k) \}^{\pm 1}$$
$$\Omega_6 = \{ \frac{1}{\sqrt{2}}(1 + i), i, \frac{1}{\sqrt{2}}(1 - k), \frac{1}{2}(1 - i - j + k) \}^{\pm 1}$$
$$\Omega_7 = \{ k \}^{\pm 1}$$
$$\Omega_8 = \{ j \}^{\pm 1}$$

Now note that the $\Omega_i$’s partition $G \setminus \{1, -1\}$. Thus, by Theorem 2.3 we can say that $\mathcal{C} := \bigcup_{i=1}^{8} Orb(C_i)$ is a $G$-regular cycle-decomposition of $K_{48} - I$. Now set $F_i = Orb_{S_i}(C_i)$
where
\[
S_i = \begin{cases} 
K & \text{for } 1 \leq i \leq 3; \\
L & \text{for } 4 \leq i \leq 6; \\
G & \text{for } i = 7, 8.
\end{cases}
\]

By Remark 2.4, each \( F_i \) is a 2-factor of \( K_{48} - I \) with \( Stab_G(F_i) = S_i \), hence \( Orb_G(F_i) \) has length 3 or 4 or 1 according to whether \( 1 \leq i \leq 3 \) or \( 4 \leq i \leq 6 \) or \( i = 7, 8 \), respectively. The cycles of \( F_i \) are triangles or quadrangles according to whether or not \( i \leq 3 \). Thus, recalling that \( C \) is a cycle-decomposition of \( K_{48} - I \), we conclude that \( \mathcal{F} := \bigcup_{i=1}^{8} Orb_G(F_i) \) is a \( G \)-regular 2-factorization of \( K_{48} - I \) with 9 triangle-factors and 14 quadrangle-factors, namely a \( G \)-regular solution of HWP(48; 3, 4; 9, 14).

### 3.4 An octahedral solution of HWP(48; 3, 4; 13, 10)

Consider the nine cycles of \( Cycle(G) \) defined as follows.

\[
\begin{align*}
C_1 &= (1, -\frac{1}{\sqrt{2}}(i + j), -\frac{1}{\sqrt{2}}(1 + j)) \\
C_2 &= (1, \frac{1}{2}(1 - i + j - k), -\frac{1}{\sqrt{2}}(i + k)) \\
C_3 &= (1, \frac{1}{\sqrt{2}}(-i + j), \frac{1}{2}(1 - i - j - k)) \\
C_4 &= (1, \frac{1}{2}(-1 + i - j + k), \frac{1}{\sqrt{2}}(i - k)) \\
C_5 &= (1, \frac{1}{2}(-1 - i + j + k), \frac{1}{2}(-1 + i - j - k)) \\
C_6 &= (1, k, -1, -k) \\
C_7 &= (1, j, -1, -j) \\
C_8 &= (1, -\frac{1}{2}(1 + i + j + k), \frac{1}{2}(-1 + i - j + k), \frac{1}{\sqrt{2}}(1 + j)) \\
C_9 &= (1, -\frac{1}{\sqrt{2}}(1 + k), -k, \frac{1}{2}(-1 + i + j - k))
\end{align*}
\]

We note that \( Stab(C_i) = V(C_i) \) for \( 5 \leq i \leq 7 \) while all other \( C_i \)'s have trivial \( G \)-stabilizer. Thus, by Lemma 2.2, one can check that \( Orb(C_i) \) is a \( \ell_i \)-cycle decomposition of \( Cay[G : \Omega_i] \) where \( \ell_i \) is the length of \( C_i \) and where the \( \Omega_i \)'s are the symmetric subsets of \( G \) listed below.

\[
\begin{align*}
\Omega_1 &= \left\{-\frac{1}{\sqrt{2}}(i + j), -\frac{1}{\sqrt{2}}(1 + j), \frac{1}{2}(1 + i + j - k)\right\}^{\pm 1} \\
\Omega_2 &= \left\{\frac{1}{2}(1 - i + j - k), -\frac{1}{\sqrt{2}}(i + k), \frac{1}{\sqrt{2}}(1 + i)\right\}^{\pm 1} \\
\Omega_3 &= \left\{\frac{1}{\sqrt{2}}(-i + j), \frac{1}{2}(1 - i - j - k), \frac{1}{\sqrt{2}}(j - k)\right\}^{\pm 1} \\
\Omega_4 &= \left\{\frac{1}{2}(-1 + i - j + k), \frac{1}{\sqrt{2}}(i - k), -\frac{1}{\sqrt{2}}(j + k)\right\}^{\pm 1} \\
\Omega_5 &= \left\{\frac{1}{2}(-1 - i + j + k)\right\}^{\pm 1} \\
\Omega_6 &= \left\{k\right\}^{\pm 1} \\
\Omega_7 &= \left\{j\right\}^{\pm 1} \\
\Omega_8 &= \left\{-\frac{1}{2}(1 + i + j + k), i, \frac{1}{\sqrt{2}}(-1 + i), \frac{1}{\sqrt{2}}(1 + j)\right\}^{\pm 1} \\
\Omega_9 &= \left\{-\frac{1}{\sqrt{2}}(1 + k), \frac{1}{2}(1 - k), \frac{1}{2}(1 - i + j + k), \frac{1}{2}(-1 + i + j - k)\right\}^{\pm 1}
\end{align*}
\]
Now note that the $\Omega_i$’s partition $G \setminus \{1, -1\}$. Thus, by Theorem 2.3 we can say that $\mathcal{C} := \bigcup_{i=1}^{9} Orb(C_i)$ is a $G$-regular cycle-decomposition of $K_{48} - I$. Now set $F_i = Orb_{S_i}(C_i)$ where

$$S_i = \begin{cases} K & \text{for } 1 \leq i \leq 4; \\ G & \text{for } 5 \leq i \leq 7; \\ L & \text{for } i = 8, 9. \end{cases}$$

By Remark 2.4, each $F_i$ is a 2-factor of $K_{48} - I$ with $Stab_G(F_i) = S_i$, hence $Orb_G(F_i)$ has length 3 or 1 or 4 according to whether $1 \leq i \leq 4$ or $5 \leq i \leq 7$ or $i = 8, 9$, respectively.

The cycles of $F_i$ are triangles or quadrangles according to whether or not $i \leq 5$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{9} Orb_G(F_i)$ is a $G$-regular 2-factorization of $K_{48} - I$ with 13 triangle-factors and 10 quadrangle-factors, namely a $G$-regular solution of HWP$(48; 3, 4; 13, 10)$.

### 3.5 An octahedral solution of HWP$(48; 3, 4; 15, 8)$

Consider the seven cycles of $Cycle(G)$ defined as follows.

$$C_1 = \left\{1, \left\{ -1 - i + j + k, \frac{1}{\sqrt{2}}(i + k) \right\} \right\}$$

$$C_2 = \left\{1, -\frac{1}{\sqrt{2}}(i + j), \frac{1}{\sqrt{2}}(1 + j) \right\}$$

$$C_3 = \left\{1, \frac{1}{2}(-1 + i + j + k), \frac{1}{2}(1 - i + j + k) \right\}$$

$$C_4 = \left\{1, \frac{1}{2}(1 + i + j + k), \frac{1}{\sqrt{2}}(1 + j) \right\}$$

$$C_5 = \left\{1, \frac{1}{2}(1 - i + j - k), \frac{1}{\sqrt{2}}(i - k) \right\}$$

$$C_6 = \left\{1, -j, k, -\frac{1}{\sqrt{2}}(1 - k) \right\}$$

$$C_7 = \left\{1, \frac{1}{\sqrt{2}}(i - j), \frac{1}{2}(-1 - i + j - k), \frac{1}{2}(-1 + i + j + k) \right\}$$

Here, every $C_i$ has trivial stabilizer. Thus, by Lemma 2.2, one can check that $Orb(C_i)$ is a $\ell_i$-cycle decomposition of $\text{Cay}[G: \Omega_i]$ where $\ell_i$ is the length of $C_i$ and where the $\Omega_i$’s are the symmetric subsets of $G$ listed below.

$$\Omega_1 = \{ \frac{1}{2}(-1 - i + j + k), \frac{1}{\sqrt{2}}(i + k), \frac{1}{\sqrt{2}}(-j + k) \}^{\pm 1}$$

$$\Omega_2 = \{ -\frac{1}{\sqrt{2}}(i + j), -\frac{1}{\sqrt{2}}(1 + j), \frac{1}{2}(1 + i + j - k) \}^{\pm 1}$$

$$\Omega_3 = \{ \frac{1}{2}(-1 + i + j - k), \frac{1}{2}(1 - i + j + k), \frac{1}{2}(-1 - i + j - k) \}^{\pm 1}$$

$$\Omega_4 = \{ \frac{1}{2}(1 + i + j + k), \frac{1}{\sqrt{2}}(1 + j), \frac{1}{\sqrt{2}}(1 + i) \}^{\pm 1}$$

$$\Omega_5 = \{ \frac{1}{2}(1 - i + j - k), \frac{1}{2}(i - k), \frac{1}{2}(j + k) \}^{\pm 1}$$

$$\Omega_6 = \{ -j, +i, \frac{1}{\sqrt{2}}(1 - k), -\frac{1}{\sqrt{2}}(1 - k) \}^{\pm 1}$$

$$\Omega_7 = \{ \frac{1}{\sqrt{2}}(i - j), -\frac{1}{\sqrt{2}}(1 + i), +k, \frac{1}{2}(-1 + i + j + k) \}^{\pm 1}$$

Now note that the $\Omega_i$’s partition $G \setminus \{1, -1\}$. Thus, by Theorem 2.3 we can say that $\mathcal{C} := \bigcup_{i=1}^{7} Orb(C_i)$ is a $G$-regular cycle-decomposition of $K_{48} - I$. Set $F_i = Orb_{S_i}(C_i)$ where

$$S_i = \begin{cases} K & \text{for } 1 \leq i \leq 5; \\ L & \text{for } i = 6, 7. \end{cases}$$
By Remark 2.4, each $F_i$ is a 2-factor of $K_{48} - I$ with $\text{Stab}_G(F_i) = S_i$, hence $\text{Orb}_G(F_i)$ has length 3 or 4 according to whether $1 \leq i \leq 5$ or $i = 6, 7$, respectively. The cycles of $F_i$ are triangles or quadrangles according to whether or not $i \leq 5$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{48} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{10} \text{Orb}_G(F_i)$ is a $G$-regular 2-factorization of $K_{48} - I$ with 15 triangle-factors and 8 quadrangle-factors, namely a $G$-regular solution of HWP$(48; 3, 4; 15, 8)$.

### 3.6 An octahedral solution of HWP$(48; 3, 4; 17, 6)$

Consider the ten cycles of $\text{Cycle}(G)$ defined as follows.

$$
\begin{align*}
C_1 &= (1, -\frac{1}{\sqrt{2}}(1 - k), -\frac{1}{\sqrt{2}}(i + k)) \\
C_2 &= (1, -\frac{1}{\sqrt{2}}(i + j), \frac{1}{2}(-1 + i + j + k)) \\
C_3 &= (1, \frac{1}{2}(1 + i - j - k), -\frac{1}{\sqrt{2}}(1 + j)) \\
C_4 &= (1, \frac{1}{\sqrt{2}}(-i + j), \frac{1}{\sqrt{2}}(-i + k)) \\
C_5 &= (1, \frac{1}{2}(1 - i + j - k), \frac{1}{\sqrt{2}}(1 - j)) \\
C_6 &= (1, \frac{1}{2}(-1 - i + j + k), \frac{1}{2}(-1 + i - j - k)) \\
C_7 &= (1, \frac{1}{2}(-1 + i + j - k), \frac{1}{2}(-1 - i - j + k)) \\
C_8 &= (1, k, -1, -k) \\
C_9 &= (1, j, -1, -j) \\
C_{10} &= (1, \frac{1}{\sqrt{2}}(1 + i), \frac{1}{\sqrt{2}}(1 - i), \frac{1}{2}(1 - i - j + k))
\end{align*}
$$

We note that $\text{Stab}(C_i) = V(C_i)$ for $6 \leq i \leq 9$ while all other $C_i$’s have trivial stabilizer. Thus, by Lemma 2.2, one can check that $\text{Orb}(C_i)$ is a $\ell_i$-cycle decomposition of $\text{Cay}[G : \Omega_i]$ where $\ell_i$ is the length of $C_i$ and where the $\Omega_i$’s are the symmetric subsets of $G$ listed below.

$$
\begin{align*}
\Omega_1 &= \{-\frac{1}{\sqrt{2}}(1 - k), -\frac{1}{\sqrt{2}}(i + k), \frac{1}{2}(-1 - i + j - k)\}^{\pm 1} \\
\Omega_2 &= \{-\frac{1}{\sqrt{2}}(i + j), \frac{1}{2}(-1 + i + j + k), \frac{1}{\sqrt{2}}(-1 + i)\}^{\pm 1} \\
\Omega_3 &= \{(\frac{1}{2}(1 + i - j - k), -\frac{1}{\sqrt{2}}(1 + j), \frac{1}{\sqrt{2}}(j + k))\}^{\pm 1} \\
\Omega_4 &= \{\frac{1}{\sqrt{2}}(-i + j), \frac{1}{\sqrt{2}}(-i + k), \frac{1}{2}(1 - i - j - k)\}^{\pm 1} \\
\Omega_5 &= \{(\frac{1}{2}(1 - i + j - k), \frac{1}{\sqrt{2}}(1 - j), \frac{1}{\sqrt{2}}(j - k))\}^{\pm 1} \\
\Omega_6 &= \{\frac{1}{2}(-1 - i + j + k)\}^{\pm 1} \\
\Omega_7 &= \{\frac{1}{2}(-1 + i + j - k)\}^{\pm 1} \\
\Omega_8 &= \{k\}^{\pm 1} \\
\Omega_9 &= \{j\}^{\pm 1} \\
\Omega_{10} &= \{\frac{1}{\sqrt{2}}(1 + i), i, \frac{1}{\sqrt{2}}(1 - k), \frac{1}{2}(1 - i - j + k)\}^{\pm 1}
\end{align*}
$$

Now note that the $\Omega_i$’s partition $G \setminus \{1, -1\}$. Thus, by Lemma 2.2 we can say that $\mathcal{C} := \bigcup_{i=1}^{10} \text{Orb}(C_i)$ is a $G$-regular cycle-decomposition of $K_{48} - I$. Set $F_i = \text{Orb}_{S_i}(C_i)$
where

\[ S_i = \begin{cases} 
K & \text{for } 1 \leq i \leq 5; \\
G & \text{for } 6 \leq i \leq 9; \\
L & \text{for } i = 10.
\end{cases} \]

By Remark 2.4, each \( F_i \) is a 2-factor of \( K_{48} \) with \( Stab_G(F_i) = S_i \), hence \( Orb_G(F_i) \) has length 3 or 1 or 4 according to whether \( 1 \leq i \leq 5 \) or \( 6 \leq i \leq 9 \) or \( i = 10 \), respectively. The cycles of \( F_i \) are triangles or quadrangles according to whether or not \( i \leq 7 \). Thus, recalling that \( \mathcal{C} \) is a cycle-decomposition of \( K_{48} - I \), we conclude that \( F := \bigcup_{i=1}^{10} Orb_G(F_i) \) is a \( G \)-regular 2-factorization of \( K_{48} - I \) with 17 triangle-factors and 6 quadrangle-factors, namely a \( G \)-regular solution of HWP(48; 3, 4; 17, 6).

4 Dicyclic solutions of two Hamilton-Waterloo problems

In this section \( G \) will denote the dicyclic group of order 24 which is usually denoted by \( Q_{24} \). Thus \( G \) has the following presentation:

\[ G = \langle a, b | a^{12} = 1, b^2 = a^6, b^{-1}ab = a^{-1} \rangle \]

Note that the elements of \( G \) can be written in the form \( a^ib^j \) with \( 0 \leq i \leq 11 \) and \( j = 0, 1 \). The group \( G \) has a unique involution which is \( a^6 \) and we will need to consider the following subgroups of \( G \):

- \( H = \langle b \rangle = \{1, b, a^6, a^6b\} \);
- \( K = \langle a^2 \rangle = \{1, a^2, a^4, a^6, a^8, a^{10}\} \);
- \( L = \langle a^2b, a^3 \rangle = \{1, a^3, a^6, a^9, a^2b, a^8b, a^5b, a^{11}b\} \).

4.1 A dicyclic solution of HWP(24; 3, 4; 7, 4)

Consider the four cycles of \( Cycle(G) \) defined as follows.

\[ C_1 = (1, a^3b, a^5) \]
\[ C_2 = (1, a^{10}, a^7b) \]
\[ C_3 = (1, a^4, a^8) \]
\[ C_4 = (1, b, a^3b, a) \]

We note that the \( Stab(C_3) = V(C_3) \) while all other \( C_i \)'s have trivial stabilizer. Thus, by Lemma 2.2, one can check that \( Orb(C_i) \) is a \( \ell_i \)-cycle decomposition of \( Cay[G : \Omega_i] \) where \( \ell_i \) is the length of \( C_i \) and where the \( \Omega_i \)'s are the symmetric subsets of \( G \) listed below.

\[ \Omega_1 = \{a^3b, a^5, a^2b\}^{\pm1} \]
\[ \Omega_2 = \{a^2, ab, a^5b\}^{\pm1} \]
\[ \Omega_3 = \{a^4\}^{\pm1} \]
\[ \Omega_4 = \{b, a^3, a^4b, a\}^{\pm1} \]

Now note that the \( \Omega_i \)'s partition \( G \setminus \{1, a^6\} \). Thus, by Theorem 2.3 we can say that \( \mathcal{C} := \bigcup_{i=1}^{4} Orb(C_i) \) is a \( G \)-regular cycle-decomposition of \( K_{24} - I \). Now set \( F_i = Orb_{S_i}(C_i) \)
By Remark 2.4, each \( F_i \) is a 2-factor of \( K_{24} - I \) with \( \text{Stab}_G(F_i) = S_i \), hence \( \text{Orb}_G(F_i) \) has length 3 or 1 or 4 according to whether \( i = 1, 2 \) or \( i = 3 \) or \( i = 4 \), respectively.

The cycles of \( F_i \) are triangles or quadrangles according to whether or not \( i \leq 3 \). Thus, recalling that \( \mathcal{C} \) is a cycle-decomposition of \( K_{48} - I \), we conclude that \( \mathcal{F} := \bigcup_{i=1}^4 \text{Orb}_G(F_i) \) is a \( G \)-regular 2-factorization of \( K_{24} - I \) with 7 triangle-factors and 4 quadrangle-factors, namely a \( G \)-regular solution of HWP(24; 3, 4; 9, 2).

### 4.2 A dicyclic solution of HWP(24; 3, 4; 9, 2)

Consider the four cycles of \( \text{Cycle}(G) \) defined as follows.

\[
C_1 = (1, b, a^6, a^6b) \\
C_2 = (1, a^4b, a^6, a^{10}b) \\
C_3 = (1, a^4, a^7b) \\
C_4 = (1, a^3b, a^8b) \\
C_5 = (a^4, a^7, a^5)
\]

We note that \( \text{Stab}(C_i) = V(C_i) \) for \( i = 1, 2 \) while all other \( C_i \)'s have trivial stabilizer. By Lemma 2.2, one can check that \( \text{Orb}(C_i) \) is a \( \ell_i \)-cycle decomposition of \( \text{Cay}[G : \Omega_i] \) where \( \ell_i \) is the length of \( C_i \) and where the \( \Omega_i \)'s are the symmetric subsets of \( G \) listed below.

\[
\Omega_1 = \{b\}^{\pm 1} \\
\Omega_2 = \{a^4b\}^{\pm 1} \\
\Omega_3 = \{a^4, ab, a^5b\}^{\pm 1} \\
\Omega_4 = \{a^3b, a^2b, a^5\}^{\pm 1} \\
\Omega_5 = \{a^4, a^2, a^3\}^{\pm 1}
\]

Also here the \( \Omega_i \)'s partition \( G \setminus \{1, a^6\} \), hence \( \mathcal{C} := \bigcup_{i=1}^5 \text{Orb}_G(C_i) \) is a \( G \)-regular cycle-decomposition of \( K_{24} - I \) by Theorem 2.3. Now set:

\[
F_1 = \text{Orb}_G(C_1), \quad F_2 = \text{Orb}_G(C_2), \\
F_3 = \text{Orb}_L(C_3), \quad F_4 = \text{Orb}_H(C_4) \cup \text{Orb}_H(C_5).
\]

By Remark 2.4, each \( F_i \) is a 2-factor of \( K_{24} - I \) and we have

\[
\text{Stab}_G(F_1) = \text{Stab}_G(F_2) = G; \quad \text{Stab}_G(F_3) = L; \quad \text{Stab}_G(F_4) = H
\]

so that the lengths of the \( G \)-orbits of \( F_1, \ldots, F_4 \) are 1, 1, 3 and 6, respectively. The cycles of \( F_i \) are triangles or quadrangles according to whether or not \( i \geq 3 \). Thus, recalling that \( \mathcal{C} \) is a cycle-decomposition of \( K_{48} - I \), we conclude that \( \mathcal{F} := \bigcup_{i=1}^5 \text{Orb}_G(F_i) \) is a \( G \)-regular 2-factorization of \( K_{24} - I \) with 9 triangle-factors and 2 quadrangle-factors, namely a \( G \)-regular solution of HWP(24; 3, 4; 9, 2).
5 A special linear solution of HWP(24; 3, 4; 5, 6)

In this section $G$ will denote the 2-dimensional special linear group over $\mathbb{Z}_3$, usually denoted by $SL_2(3)$, namely the group of $2 \times 2$ matrices with elements in $\mathbb{Z}_3$ and determinant one. The only involution of $G$ is $2E$ where $E$ is the identity matrix of $G$. The 2-Sylow subgroup $Q$ of $G$, isomorphic to the group of quaternions, is the following:

$$Q = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

We will also need to consider the subgroup $H$ of $G$ of order 6 generated by the matrix $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$. Hence we have:

$$H = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \right\}.$$

The use of the special linear group $G$ was crucial in [5] to get a Steiner triple system of any order $v = 144n + 25$ with an automorphism group acting sharply transitively an all but one point. Here $G$ will be used to get a $G$-regular solution of the last Hamilton-Waterloo problem left open in [11].

Consider the six cycles of $\text{Cycle}(G)$ defined as follows.

$$C_1 = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \right)$$

$$C_2 = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} \right)$$

$$C_3 = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \right)$$

$$C_4 = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \right)$$

$$C_5 = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} \right)$$

$$C_6 = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)$$

Here the stabilizer of $C_i$ is trivial for $i = 1, 6$ while it coincides with $V(C_i)$ for $2 \leq i \leq 5$. By Lemma 2.2, one can check that $\text{Orb}(C_i)$ is a $\ell_i$-cycle decomposition of $\text{Cay}[G : \Omega_i]$ where $\ell_i$ is the length of $C_i$ and where the $\Omega_i$’s are the symmetric subsets of $G$ listed below.

$$\Omega_1 = \left\{ \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \right\}^{\pm 1}$$

$$\Omega_2 = \left\{ \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix} \right\}^{\pm 1}$$

$$\Omega_3 = \left\{ \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} \right\}^{\pm 1}$$

$$\Omega_4 = \left\{ \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}^{\pm 1}$$

$$\Omega_5 = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \right\}^{\pm 1}$$

$$\Omega_6 = \left\{ \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}^{\pm 1}$$
Once again we see that the $\Omega_i$’s partition $G \setminus \{E, 2E\}$, therefore $\mathcal{C} := \bigcup_{i=1}^{6} \text{Orb}(C_i)$ is a $G$-regular cycle-decomposition of $K_{24} - I$. Now set $F_i = \text{Orb}_{S_i}(C_i)$ with

$$S_i = \begin{cases} Q & \text{for } i = 1; \\ G & \text{for } 2 \leq i \leq 5; \\ H & \text{for } i = 6. \end{cases}$$

By Remark 2.4, each $F_i$ is a 2-factor of $K_{24} - I$ and we have $\text{Stab}_G(F_i) = S_i$ so that the lengths of the $G$-orbits of $F_1, \ldots, F_6$ are 3, 1, 1, 1, 1 and 4, respectively.

The cycles of $F_i$ have length 3 or 4 according to whether or not $i \leq 3$. Thus, recalling that $\mathcal{C}$ is a cycle-decomposition of $K_{24} - I$, we conclude that $\mathcal{F} := \bigcup_{i=1}^{6} \text{Orb}_{G}(F_i)$ is a $G$-regular 2-factorization of $K_{24} - I$ with 5 triangle-factors and 6 quadrangle-factors, namely a $G$-regular solution of HWP$(24; 3; 4; 5; 6)$.

References


