On the cover of this journal


Given a group $A$ and a generating set $X$, the Cayley (color) graph $C(A, X)$ has the elements of $A$ as vertices and edges (colored $x$) directed from $a$ to $ax$ for each $a \in A$ and $x \in X$. Such graphs were introduced by Sir Arthur Cayley in 1878 to give a picture of a group. This viewpoint of a group as a geometric object has proved remarkably fruitful, creating a strand of thought than can be traced from Burnside, to Dehn, to Stallings, to Gromov, to Thurston, to the present field of geometric group theory. From the very beginning, one thread in this strand has been that for finite groups, a drawing, or embedding, of a Cayley graph in a surface may be useful from the viewpoint of generators and relations for the group, and the smaller the genus, the fewer relators. The smallest possible genus, over all Cayley graphs for the group $A$, is called the genus of $A$, a term introduced by Art White in 1972.

The sculpture you see on the cover of this journal shows an embedding in the surface of genus two of $C(A, \{x, y, z\})$ where

$$A = \langle x, y, z : x^2 = y^2 = z^2 = 1, (xy)^2 = (yz)^3 = (xz)^8 = 1, [y, (xz)^4] = 1 \rangle.$$ 

The $x$-edges are colored green, the $y$-edges are colored red, and the $z$-edges are colored yellow. There are six faces of 16 alternating green and yellow sides corresponding to $(xz)^8 = 1$, one you can see at the join of the two handles, one you cannot see on the other side of the join, and the other four are wrapped around the two handles. There are 16 faces of six alternating red and yellow sides corresponding to $(yz)^3 = 1$, eight you can see on the join and eight you can’t. There are 24 faces of four alternating green and red sides corresponding to $(xy)^2 = 1$. The final relator $[y, (xz)^4] = 1$ is not a face but rather a cycle twisting around a handle. These numbers are determined by the order of the group $A$, which is easily shown to be 96, since the subgroup of order two generated by $(xz)^4$ is clearly normal with quotient the group of symmetries of the cube. Thomas Tucker showed in 1984 that this is the only group of genus two.

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