Finite two-distance-transitive graphs of valency 6

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Abstract

A non-complete graph Γ is said to be \((G, 2)\)-distance-transitive if, for \(i = 1, 2\) and for any two vertex pairs \((u_1, v_1)\) and \((u_2, v_2)\) with \(d_\Gamma(u_1, v_1) = d_\Gamma(u_2, v_2) = i\), there exists \(g \in G\) such that \((u_1, v_1)^g = (u_2, v_2)\). This paper classifies the family of \((G, 2)\)-distance-transitive graphs of valency 6 which are not \((G, 2)\)-arc-transitive.

Keywords: 2-Distance-transitive graph, 2-arc-transitive graph, permutation group.

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1 Introduction

In this paper, all graphs are finite, simple, connected and undirected. For a graph Γ, we use \(V(\Gamma)\) and \(\text{Aut}(\Gamma)\) to denote its vertex set and automorphism group, respectively. For the group theoretic terminology not defined here we refer the reader to [4, 8, 26]. Let \(u, v \in V(\Gamma)\). Then the distance between \(u, v\) in \(\Gamma\) is denoted by \(d_\Gamma(u, v)\). A non-complete graph \(\Gamma\) is said to be \((G, 2)\)-distance-transitive, if for \(i = 1, 2\) and for any two vertex pairs \((u_1, v_1)\) and \((u_2, v_2)\) with \(d_\Gamma(u_1, v_1) = d_\Gamma(u_2, v_2) = i\), there exists \(g \in G\) such that \((u_1, v_1)^g = (u_2, v_2)\). An arc is an ordered pair of adjacent vertices. A vertex triple \((u, v, w)\) with \(v\) adjacent to both \(u\) and \(w\) is called a 2-arc if \(u \neq w\). The graph \(\Gamma\) is said to be \((G, 2)\)-arc-transitive if \(G\) is transitive on both the set of arcs and the set of 2-arcs.

The first remarkable result about \((G, 2)\)-arc-transitive graphs comes from Tutte [20, 21], and since then, this family of graphs has been studied extensively, see [1, 12, 15, 16, 17, 23, 24]. By definition, every non-complete \((G, 2)\)-arc-transitive graph is \((G, 2)\)-distance-transitive. The converse is not necessarily true. If a \((G, 2)\)-distance-transitive graph has

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girth 3 (length of the shortest cycle is 3), then this graph is not \((G,2)\)-arc-transitive. Thus, the family of non-complete \((G,2)\)-arc-transitive graphs is properly contained in the family of \((G,2)\)-distance-transitive graphs. The graph in Figure 1 is the Kneser graph \(KG_{6,2}\) which is \((G,2)\)-distance-transitive but not \((G,2)\)-arc-transitive of valency 6 for \(G = \text{Aut}(KG_{6,2})\). Therefore the following problem naturally arises: characterize the family of \((G,2)\)-distance-transitive graphs. At the moment, Corr, Schneider and the first author are investigating such graphs, and they classified the family of \((G,2)\)-distance-transitive but not \((G,2)\)-arc-transitive graphs of valency at most 5 in [6]. Hence 6 is the next smallest valency for \((G,2)\)-distance-transitive graphs to investigate. Our main theorem gives a classification of such graphs.

Remark 1.1. Let \(\Gamma\) be a connected \((G,2)\)-distance-transitive graph. If \(\Gamma\) has girth at least 5, then for any two vertices \(u, v\) with \(d\Gamma(u, v) = 2\), there exists a unique 2-arc between \(u\) and \(v\). Hence \(\Gamma\) is \((G,2)\)-distance-transitive implies that it is \((G,2)\)-arc-transitive. If \(\Gamma\) has girth 4, then \(\Gamma\) can be \((G,2)\)-distance-transitive but not \((G,2)\)-arc-transitive. There are infinitely many such graphs. For instance, let \(\Gamma\) be the complement of the \((2 \times p^k)\)-grid where \(p\) is a prime, and let \(M = \mathbb{Z}_p^k : \mathbb{Z}_{p^k-1}, G = \mathbb{Z}_2 \times M\). Then \(\Gamma\) is \((G,2)\)-distance-transitive but not \((G,2)\)-arc-transitive of valency \(p^k - 1\) and girth 4. There are also infinitely many \((G,2)\)-distance-transitive graphs of girth 4 that are \((G,2)\)-arc-transitive, for example the complete bipartite graphs \(K_{m,n}\). If \(\Gamma\) has girth 3, then since \(\Gamma\) is non-complete, it follows that \(G_a\) is not 2-transitive on \(\Delta(u)\), hence it is not \((G,2)\)-arc-transitive.

The line graph \(L(\Gamma)\) of a graph \(\Gamma\) has the set of edges of \(\Gamma\) as its vertex set, and two edges are adjacent in \(L(\Gamma)\) if and only if they have a common vertex in \(\Gamma\). The line graph of a complete bipartite graph \(K_{m,n}\) is called an \((m \times n)\)-grid. Let \(\Gamma\) be a connected graph. The complement graph \(\overline{\Gamma}\) of \(\Gamma\) is the graph with vertex \(V(\Gamma)\), and two vertices are adjacent in \(\overline{\Gamma}\) if and only if they are not adjacent in \(\Gamma\). The Hamming graph \(H(d,n)\) has vertex set \(\mathbb{Z}_n^d = \mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n\), and two vertices are adjacent in \(\overline{\Gamma}\) if and only if they have exactly one different coordinate. We denote by \(K_{m|b}\) the complete multipartite graph with \(m\) parts, and each part has \(b\) vertices where \(m \geq 3, b \geq 2\). Let \(p\) be a prime such that \(p \equiv 1 (\text{mod } 4)\). Then, the Paley graph \(P(p)\) is the Cayley graph \(\text{Cay}(T, S)\) for the additive group \(T = F_p^+\) with \(S = \{w^2, w^4, \ldots, w^{p-1} = 1\}\) and \(\Gamma_2(1) = \{w, w^3, \ldots, w^{p-2}\}\), where \(w\) is a primitive element of \(F_p\), and \(\text{Aut}(\Gamma) \cong \mathbb{Z}_p : \mathbb{Z}_{p-1}\). In particular, Hamming graphs and Paley graphs are \((G,2)\)-distance-transitive for \(G = \text{Aut}(\Gamma)\), see [3, 13].
The diameter $\text{diam}(\Gamma)$ of a graph $\Gamma$ is the maximum distance occurring over all pairs of vertices. Let $u \in V(\Gamma)$ and $i = 1, 2, \ldots$, $\text{diam}(\Gamma)$. We use $\Gamma_i(u)$ to denote the set of vertices at distance $i$ with vertex $u$ in $\Gamma$. Sometimes, $\Gamma_1(u)$ is also denoted by $\Gamma(u)$. Let $\Omega$ be a set of cardinality $n$. Then the Knörsen graph $KG_{n,k}$ is the graph with vertex set all $k$-subsets of $\Omega$, and two $k$-subsets are adjacent if and only if they are disjoint. The triangular graph $T(n)$ is the graph with vertex set all 2-subsets of $\Omega$, and two 2-subsets are adjacent if and only if they share one common element. Thus $KG_{n,2} = T(n)$. A subgraph $X$ of $\Gamma$ is an induced subgraph if two vertices of $X$ are adjacent in $\Gamma$ if and only if they are adjacent in $\Gamma$. When $U \subseteq V(\Gamma)$, we use $[U]$ to denote the subgraph of $\Gamma$ induced by $U$.

Since complete graphs have diameter 1, they do not provide interesting examples. Our main theorem determines the family of non-complete $(G, 2)$-distance-transitive graphs of valency 6 which are not $(G, 2)$-arc-transitive.

**Theorem 1.2.** Let $\Gamma$ be a connected non-complete $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive graph of valency 6. Let $u \in V(\Gamma)$. Then one of the following holds.

1. $\Gamma$ has girth 4, and $(\Gamma, G) = ((\overline{2} \times 7) - \text{grid}, S_2 \times M)$ where $M$ is a 2-transitive but not 3-transitive subgroup of $S_7$.
2. $[\Gamma(u)]$ is connected, and $\Gamma$ is isomorphic to one of: $T(5)$, Paley graph $P(13)$, $K_3[3]$ or $K_4[2]$.
3. $[\Gamma(u)]$ is disconnected, and either
   (3.1) $[\Gamma(u)] \cong 2K_3$, $\Gamma \cong H(2, 4)$, or $|\Gamma_2(u)| = 18$ and $\Gamma$ is a line graph; or
   (3.2) $[\Gamma(u)] \cong 3K_2$, $\Gamma \cong KG_{6,2}$, or $|\Gamma_2(u)| = 12, 24$.

**Remark 1.3.** (1) There exist graphs $\Gamma$ in Theorem 1.2 (3.1) such that $|\Gamma_2(u)| = 18$. For instance the generalized hexagon of order $(3, 1)$ and the generalized dodecagon of order $(3, 1)$. These two graphs are locally isomorphic to $2K_3$ and $|\Gamma_2(u)| = 18$. By [3, p.223], they are $(G, 2)$-distance-transitive for $G = \text{Aut}(\Gamma)$, since they are non-complete and have girth 3, they are not $(G, 2)$-arc-transitive.

(2) There exist graphs $\Gamma$ in Theorem 1.2 (3.2) such that $|\Gamma_2(u)| = 12$ and also exist graphs such that $|\Gamma_2(u)| = 24$. For instance $H(3, 3)$ has valency 6, $[\Gamma(u)] \cong 3K_2$ and $|\Gamma_2(u)| = 12$; the halved foster graph has valency 6, $[\Gamma(u)] \cong 3K_2$ and $|\Gamma_2(u)| = 24$. By [3, p.223], these two graphs are $(G, 2)$-distance-transitive for $G = \text{Aut}(\Gamma)$, since they are non-complete and have girth 3, they are not $(G, 2)$-arc-transitive.

**2 Proof of Theorem 1.2**

In this section, we will prove our main theorem by a series of lemmas. All graphs are non-complete graphs.

A graph $\Gamma$ is said to be $G$-distance-transitive if $G$ is transitive on the ordered pairs of vertices at any given distance. The study of finite $G$-distance-transitive graphs goes back to Higman’s paper [10] in which “groups of maximal diameter” were introduced. These are permutation groups $G$ which act distance-transitively on some graph. Then $G$-distance-transitive graphs have been studied extensively and a classification is almost done, see [2, 9, 11, 18, 19, 22, 25]. By definition, every non-complete $G$-distance-transitive graph is $(G, 2)$-distance-transitive.

The following remark gives an useful observation.

**Remark 2.1.** Let $\Gamma$ be a $(G, 2)$-distance-transitive graph. Let $u, w$ be two vertices such that $d_\Gamma(u, w) = 2$. 

Suppose that \(|\Gamma_3(u) \cap \Gamma(w)| = 0\). Then since \(\Gamma\) is \((G,2)\)-distance-transitive, \(\Gamma\) has diameter 2 and so it is \(G\)-distance-transitive.

Suppose that \(|\Gamma_3(u) \cap \Gamma(w)| = 1\). Let \((u_0, \ldots, u_i)\) be a path with \(d_\Gamma(u_0, u_i) = i\) where \(i = \text{diam}(\Gamma)\). Then for each \(j \leq \text{diam}(\Gamma) - 2\), \(|\Gamma_3(u_j) \cap \Gamma(u_{j+2})| = 1\). Note that, \(\Gamma_{j+3}(u_0) \cap \Gamma(u_{j+2}) \subseteq \Gamma_3(u_j) \cap \Gamma(u_{j+2})\), and so \(|\Gamma_{j+3}(u_0) \cap \Gamma(u_{j+2})| = 1\), hence \(\Gamma\) is also \(G\)-distance-transitive.

We use \(G_u^{[1]}\) to denote the kernel of the \(G_u\)-action on \(\Gamma(u)\).

**Lemma 2.2.** Let \(\Gamma\) be a \((G,2)\)-distance-transitive graph. Let \(u, w \in V(\Gamma)\) be such that \(d_\Gamma(u, w) = 2\). Let \(g \in G_u^{[1]}\) be with order a prime \(p\). Suppose that \(|\Gamma_3(u) \cap \Gamma(w)| < p\). Then \(g\) is not trivial on \(\Gamma_2(u)\).

**Proof.** Suppose that \(g\) is trivial on \(\Gamma_2(u)\). Let \(w_i \in \Gamma_2(u)\). Since \(g \in G_u^{[1]}\) and \(g\) is trivial on \(\Gamma_2(u)\), \(g\) fixes all the vertices in \((\Gamma(u) \cup \Gamma_2(u)) \cap \Gamma(w_i)\) and \(g \in G_{w_i}\). In particular, \(g\) fixes \(\Gamma_3(u) \cap \Gamma(w_i)\) setwise.

Since \(\Gamma\) is \((G,2)\)-distance-transitive and \(|\Gamma_3(u) \cap \Gamma(w)| < p\), \(|\Gamma_3(u) \cap \Gamma(w_i)| < p\). Since the order of \(g\) is prime \(p\) and \(g\) fixes \(\Gamma_3(u) \cap \Gamma(w_i)\) setwise, it follows that \(g\) fixes all the vertices in \(\Gamma_3(u) \cap \Gamma(w_i)\). Thus \(g \in G_u^{[1]}\). Since \(w_i\) is any vertex of \(\Gamma_2(u)\), \(g\) fixes all the vertices of \(\Gamma_3(u)\). For any \(v \in \Gamma(u)\), \(\Gamma_2(v) \subseteq \Gamma(u) \cap \Gamma_2(u) \cap \Gamma_3(u)\). Thus \(g \in G_v^{[1]}\) and fixes all the vertices of \(\Gamma_2(v)\).

Since \(\Gamma\) is \((G,2)\)-distance-transitive, for any \(z \in \Gamma_2(v)\), \(|\Gamma_3(v) \cap \Gamma(z)| < p\). Since \(g\) fixes all the vertices in \((\Gamma(v) \cup \Gamma_2(v)) \cap \Gamma(z)\), \(g\) fixes all the vertices in \(\Gamma_3(v) \cap \Gamma(z)\). Thus \(g \in G_z^{[1]}\). In particular, \(g\) fixes all the vertices of \(\Gamma_4(u)\). Since \(\Gamma\) is connected, by induction, \(g\) fixes all the vertices of \(\Gamma\), so \(g = 1\), which is a contradiction. Thus \(g\) is not trivial on \(\Gamma_2(u)\).

**Lemma 2.3.** Let \(\Gamma\) be a \((G,2)\)-distance-transitive graph of valency 6. Let \(u, w \in V(\Gamma)\) be such that \(d_\Gamma(u, w) = 2\). If \(\Gamma\) has girth 4 and \(|\Gamma(u) \cap \Gamma(w)| = 3\), then \(\Gamma\) is \((G,2)\)-arc-transitive.

**Proof.** Suppose that \(\Gamma\) has girth 4 and \(|\Gamma(u) \cap \Gamma(w)| = 3\). Let \((u, v, w)\) be a 2-arc. Then \(d_\Gamma(u, w) = 2\) and \(|\Gamma_2(u) \cap \Gamma(v)| = 5\). Since \(\Gamma\) is \((G,2)\)-distance-transitive, there are 30 edges between \(\Gamma(u)\) and \(\Gamma_2(u)\). Since \(|\Gamma(u) \cap \Gamma(w)| = 3\) and \(|\Gamma(u) \cap \Gamma(w)| \cap |\Gamma_2(u)| = 30\), it follows that \(|\Gamma_2(u)| = 10\). Again since \(\Gamma\) is \((G,2)\)-distance-transitive, \(G_u\) is transitive on both \(\Gamma(u)\) and \(\Gamma_2(u)\), so both \(|\Gamma(u)|\) and \(|\Gamma_2(u)|\) divide \(|G_u|\), hence 30 divides \(|G_u|\). Thus 5 divides \(|G_{u,v}|\), so \(G_{u,v}\) has an element \(g\) of order 5. Therefore either \(g\) is regular on \(\Gamma(u)\) \(\setminus\) \(\{v\}\) or is trivial on \(\Gamma(u) \setminus \{v\}\). If \(g\) is regular on \(\Gamma(u) \setminus \{v\}\), then \(G_{u,v}\) is transitive on \(\Gamma(u) \setminus \{v\}\), so \(G_u\) is 2-transitive on \(\Gamma(u)\). Thus \(\Gamma\) is \((G,2)\)-arc-transitive.

Now suppose that \(g\) is trivial on \(\Gamma(u) \setminus \{v\}\). Then \(g \in G_u^{[1]}\). Since \(|\Gamma(u) \cap \Gamma(w)| = 3\), it follows that \(|\Gamma_3(u) \cap \Gamma(w)| \leq 3 < 5\). Thus by Lemma 2.2, \(g\) is not trivial on \(\Gamma_2(u)\). Hence \(g\) has orbits of size 5 on \(\Gamma_2(u)\). Since \(g\) fixes \(\Gamma_2(u) \cap \Gamma(v_i)\) setwise and \(|\Gamma_2(u) \cap \Gamma(v_i)| = 5\), it follows that \(g\) is transitive on \(\Gamma_2(u) \cap \Gamma(v_i)\). Thus \(G_{u,v_i}\) is transitive on \(\Gamma_2(u) \cap \Gamma(v_i)\), so \(\Gamma\) is \((G,2)\)-arc-transitive.

**Lemma 2.4.** ([6]) Let \(\Gamma \cong K_{m,m}\) with \(m \geq 2\). Then \(\Gamma\) is \((G,2)\)-distance-transitive if and only if it is \((G,2)\)-arc-transitive.

A permutation group \(G\) on a set \(\Omega\) is said to be 2-homogeneous, if \(G\) is transitive on the set of 2-subsets of \(\Omega\).
Lemma 2.5. ([8, Theorem 9.4B]) Let $G$ be a 2-homogeneous permutation group which is not $2$-transitive of degree $n$. Then $n = p^e \equiv 3 \pmod{4}$ where $p$ is a prime.

Lemma 2.6. Let $\Gamma$ be a $(G,2)$-distance-transitive but not $(G,2)$-arc-transitive graph of valency 6. If $\Gamma$ has girth 4, then $(\Gamma,G) = ((2 \times 7)$-grid, $S_2 \times M)$ where $M$ is a 2-transitive but not 3-transitive subgroup of $S_7$.

Proof. Suppose that $\Gamma$ has girth 4. Let $(u,v,w)$ be a 2-arc. Then $d_\Gamma(u,w) = 2$. If $|\Gamma_2(u) \cap \Gamma(v)| = 5$ and $|\Gamma(u) \cap \Gamma(w)| \geq 2$. Further there are 30 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Since $\Gamma$ is $(G,2)$-distance-transitive, $|\Gamma(u) \cap \Gamma(w)|$ divides 30. Since $2 \leq |\Gamma(u) \cap \Gamma(w)| \leq 6$, we have $|\Gamma(u) \cap \Gamma(w)| = 2, 3, 5$ or $6$.

Suppose first that $|\Gamma(u) \cap \Gamma(w)| = 2$. Then since $\Gamma$ has girth 4, each 2-arc of $\Gamma$ lies in a unique 4-cycle. Thus, there is a 1-1 mapping between the unordered vertex pairs in $\Gamma(u)$ and vertices in $\Gamma_2(u)$. Since $G_u$ is transitive on $\Gamma_2(u)$, it follows that $G_u$ is transitive on the set of unordered vertex pairs in $\Gamma(u)$. Hence $G_u^{\Gamma(u)}$ is 2-homogeneous on $\Gamma(u)$. Further, since $\Gamma$ is not $(G,2)$-arc-transitive, $G_u^{\Gamma(u)}$ is not 2-transitive on $\Gamma(u)$. Thus by Lemma 2.5, the valency of $\Gamma$ is $p^e \equiv 3 \pmod{4}$ where $p$ is a prime, contradicting the fact that $\Gamma$ has valency 6.

Next, if $|\Gamma(u) \cap \Gamma(w)| = 3$, then by Lemma 2.3, $\Gamma$ is $(G,2)$-arc-transitive, which is a contradiction.

Thirdly, suppose that $|\Gamma(u) \cap \Gamma(w)| = 5$. Then $|\Gamma_3(u) \cap \Gamma(w)| \leq 1$. It follows from Remark 2.1 that $\Gamma$ is $G$-distance-transitive. By inspecting the graphs in [3, p. 222-223], $\Gamma$ is isomorphic to $(2 \times 7)$-grid. Noting that $(2 \times 7)$-grid is $(\text{Aut}(\Gamma),2)$-arc-transitive. Thus $S_2 < G < \text{Aut}(\Gamma) \cong S_2 \times S_7$. Let $G = S_2 \times M$ where $M < S_7$. Then $G_u = M_u$. Since $\Gamma$ is $(G,2)$-distance-transitive but not $(G,2)$-arc-transitive, $M_u$ is transitive but not 2-transitive on $\Gamma(u)$. Thus $M$ is a 2-transitive but not 3-transitive subgroup of $S_7$.

Finally, if $|\Gamma(u) \cap \Gamma(w)| = 6$, then $\Gamma \cong K_{6,6}$, and by Lemma 2.4, $\Gamma$ is $(G,2)$-distance-transitive implies that it is $(G,2)$-arc-transitive, which is a contradiction. □

In a non-complete graph $\Gamma$, a $2$-geodesic of $\Gamma$ is a 2-arc $(u_0, u_1, u_2)$ such that $d_\Gamma(u_0, u_2) = 2$. The graph $\Gamma$ is said to be $(G,2)$-geodesic-transitive, if $G$ is transitive on both the set of arcs and the set of 2-geodesics. Hence, a non-complete $G$-arc-transitive graph is $(G,2)$-geodesic-transitive if, for any arc $(u,v)$, $G_{u,v}$ is transitive on $\Gamma_2(u) \cap \Gamma(v)$. By definition, every $(G,2)$-geodesic-transitive graph is $(G,2)$-distance-transitive.

Suppose that $\Gamma$ is a $G$-distance-transitive graph of valency $k$ and diameter $d$. Then the cells of the distance partition with respect to vertex $u$ are orbits of $G_u$, every vertex in $\Gamma_i(u)$ is adjacent to the same number of other vertices in $\Gamma_{i-1}(u)$, say $c_i$. Similarly, every vertex in $\Gamma_i(u)$ is adjacent to the same number of other vertices in $\Gamma_{i+1}(u)$, say $b_i$. The notation $(k, b_1, \ldots, b_d; 1, c_2, \ldots, c_d)$ is called the intersection array of $\Gamma$.

Lemma 2.7. Let $\Gamma$ be a $(G,2)$-distance-transitive but not $(G,2)$-arc-transitive graph of valency 6. Let $u \in V(\Gamma)$. If $[\Gamma(u)]$ is connected, then $\Gamma$ is isomorphic to one of: $T(5)$, Paley graph $P(13)$, $K_{3[3]}$ or $K_{4[2]}$.

Proof. Suppose that $[\Gamma(u)]$ is connected. Let $(u,v,w)$ be a 2-arc such that $d_\Gamma(u,w) = 2$. Since $\Gamma$ is $(G,2)$-distance-transitive, $G_u$ is transitive on $\Gamma(u)$, so $[\Gamma(u)]$ is a vertex-transitive graph. Let $k$ be the valency of $[\Gamma(u)]$. Since $[\Gamma(u)]$ is connected and $|\Gamma(u)| = 6$, it follows that $k = 2, 3, 4, 5$. Let $\Gamma(u) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$.

If $k = 5$, then $[\Gamma(u)] \cong K_6$, and so $\Gamma \cong K_7$, contradicting the fact that $\Gamma$ is non-complete.
Suppose that \( k = 4 \). Then \( |\Gamma(u) \cap \Gamma(v_1)| = 4 \), say \( \Gamma(u) \cap \Gamma(v_1) = \{v_2, v_3, v_4, v_5\} \). Since \( |\Gamma(u) \cap \Gamma(v_6)| = 4 \) and \( v_1, v_6 \) are non-adjacent, it follows that \( \Gamma(u) \cap \Gamma(v_6) = \{v_2, v_3, v_4, v_5\} \). Thus \( |\Gamma(u)| \) has diameter 2, and \( \{v_1, v_6\} \) is a block. Since \( |\Gamma(u)| \) is vertex-transitive, \( \Gamma(u) \cong K_{3,2} \), and by [3, p.5] or [5], \( \Gamma \cong K_{3,3} \).

Suppose that \( k = 3 \). Then \( |\Gamma(u) \cap \Gamma(v_1)| = 3 \), say \( \Gamma(u) \cap \Gamma(v_1) = \{v_2, v_3, v_4\} \). Assume first that \( |\Gamma(u)| \) does not have triangles. Then every vertex of \( \{v_2, v_3, v_4\} \) is adjacent to both \( v_5 \) and \( v_6 \). Thus \( |\Gamma(u)| \cong K_{3,3} \). Then by [3, p.5] or [5], \( \Gamma \cong K_{3,3} \). Next, assume that \( |\Gamma(u)| \) has a triangle. Since \( |\Gamma(u)| \) is vertex-transitive, every vertex of \( \Gamma(u) \) lies in a triangle. Let \( (v_1, v_2, v_3) \) be a triangle. Since \( |\Gamma(u)| \) is connected, \( v_4 \) is adjacent to neither \( v_2 \) nor \( v_3 \). Thus \( v_4 \) is adjacent to both \( v_5 \) and \( v_6 \). Since \( v_4 \) lies in a triangle and \( \{v_5, v_6\} \subseteq \Gamma_2(v_1) \), it follows that \( v_5, v_6 \) are adjacent. Further, \( v_2 \) is adjacent to one of \( \{v_5, v_6\} \), say \( v_5 \), and \( v_3 \) is adjacent to the remaining vertex \( v_6 \). Thus \( |\Gamma(u)| \) is isomorphic to the 3-prism, \( (v_1, v_2, v_3) \) and \( (v_4, v_5, v_6) \) are the two triangles, and \( \{v_1, v_4\} \), \( \{v_2, v_5\} \) and \( \{v_3, v_6\} \) are edges. Since \( k = 3 \), it follows that \( |\Gamma_2(u) \cap \Gamma(v_1)| = 2 \). Set \( \Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2\} \). Then \( \Gamma(v_1) = \{u, v_2, v_3, v_4, w_1, w_2\} \). Since \( |\Gamma(v_1)| \) is isomorphic to the 3-prism, it follows that \( v_4 \) is adjacent to both \( w_1 \) and \( w_2 \), \( v_2 \) is adjacent to one of \( \{w_1, w_2\} \), say \( w_1 \), and \( v_3 \) is adjacent to \( w_2 \). Thus \( \Gamma(v_1) = \{u, v_1, v_5, v_6, w_1, w_2\} \). Since \( |\Gamma(v_1)| \) is isomorphic to the 3-prism, it follows that \( w_1 \) is adjacent to one of \( \{v_5, v_6\} \), say \( v_5 \). Thus \( \{v_1, v_2, v_4, v_5\} \subseteq \Gamma(u) \cap \Gamma(w_1) \). Since \( v_2 \in \Gamma(w_1) \), it follows that \( |\Gamma_3(u) \cap \Gamma(w_1)| \leq 1 \). Thus by Remark 2.1, \( \Gamma \) is \( G \)-distance-transitive.

Since \( \{v_1, v_2, v_4, v_5\} \subseteq \Gamma(u) \cap \Gamma(w_1) \) and \( \{v_1\} \subseteq \Gamma_2(u) \cap \Gamma(w_1) \), it follows that \( |\Gamma(u) \cap \Gamma(v_1)| = 4 \) or 5. Since \( \Gamma \) is \( (G, 2) \)-distance-transitive and \( |\Gamma_2(u) \cap \Gamma(v_1)| = 2 \), there are 12 edges between \( \Gamma(u) \) and \( \Gamma_2(u) \). Thus \( |\Gamma(u) \cap \Gamma(v_1)| \) divides 12, so \( |\Gamma(u) \cap \Gamma(w_1)| \cong K_{3,2} \). Hence \( |\Gamma_2(u)| = 3 \). Since \( G_u \) is transitive on \( \Gamma_2(u) \), \( \Gamma_2(u) \) is a vertex-transitive regular graph. Since \( w_1, w_2 \) are adjacent, \( \Gamma_2(u) \cong C_3 \). Therefore, \( |\Gamma_3(u) \cap \Gamma(w_1)| = 0 \), \( \Gamma \) has diameter 2 and has 10 vertices. In particular, the intersection array of \( \Gamma \) is \((6, 2; 1, 4)\). By inspecting the graphs in [3, p.222-223], \( \Gamma \) is \( T(5) \) (also known as the Johnson graph \( J(5, 2) \)).

If \( k = 2 \), then \( |\Gamma(u)| \cong C_6 \). Let \( (v_1, \ldots, v_6) \) be a 6-cycle. Then \( |\Gamma_2(u) \cap \Gamma(v_1)| = 3 \), and set \( \Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2, w_3\} \). Then \( \Gamma(v_1) = \{u, v_2, v_3, v_4, w_1, w_2, w_3\} \). Since \( |\Gamma(v_1)| \cong C_6 \) and \( \{v_2, u, v_6\} \) is a 2-arc, it follows that \( v_2 \) is adjacent to one of \( \{w_1, w_2, w_3\} \), say \( w_3 \), and \( w_2 \) is adjacent to both \( w_1 \) and \( w_3 \). In particular, \( v_6 \) is not adjacent to any of \( \{w_2, w_3\} \), and \( v_6 \) is not adjacent to any of \( \{w_1, w_2\} \). Since \( |\Gamma_2(u) \cap \Gamma(v_2)| = 3 \), there exist \( w_4, w_5 \in \Gamma_2(u) \) that are adjacent to \( v_2 \), and so \( \Gamma(v_2) = \{u, v_1, v_3, w_4, w_5, w_6, w_4, w_5\} \). Noting that \( |\Gamma(v_2)| \cong C_6 \) and \( \{w_1, v_1, v_3, u, v_3\} \) is a 3-arc, so \( v_3 \) is adjacent to one of \( \{w_4, w_5\} \), say \( w_5 \), \( w_1 \) is adjacent to \( w_4 \), and \( w_4, w_5 \) are adjacent. Thus, \( \{v_1, v_2, v_4, w_4\} \subseteq \Gamma(u) \cap \Gamma_2(u) \cap \Gamma(w_1) \). Hence \( |\Gamma(u) \cap \Gamma(w_1)| \leq 4 \) and \( |\Gamma_2(u) \cap \Gamma(w_1)| \geq 2 \). Since \( \Gamma \) is \( (G, 2) \)-distance-transitive and \( |\Gamma_2(u) \cap \Gamma(v_1)| = 3 \), there are 18 edges between \( \Gamma(u) \) and \( \Gamma_2(u) \). Since \( |\Gamma(u) \cap \Gamma(v_1)| \) divides 18, \( |\Gamma(u) \cap \Gamma(v_1)| = 2 \) or 3.

Suppose that \( |\Gamma(u) \cap \Gamma(v_1)| = 2 \). Then \( |\Gamma_2(u)| = 9 \). Since \( |\Gamma_2(u) \cap \Gamma(v_1)| \geq 2 \), \( |\Gamma_3(u) \cap \Gamma(w_1)| \leq 2 \). If \( |\Gamma_3(u) \cap \Gamma(w_1)| \leq 1 \), then by Remark 2.1, \( \Gamma \) is \( G \)-distance-transitive. Inspecting the graphs in [3, p.222-223], such a \( \Gamma \) does not exist. Hence \( |\Gamma_3(u) \cap \Gamma(w_1)| = 2 \). Since \( \Gamma \) is \( (G, 2) \)-distance-transitive, both \( |\Gamma(u)| \) and \( |\Gamma_2(u)| \) divide \( |G_u| \), hence 18 divides \( |G_u| \). Thus 3 divides \( |G_{u,v}| \). Therefore \( G_{u,v} \) has an element \( g \) of order 3.

Since \( |\Gamma(u) \setminus \{v\}| = 5 \), it follows that \( g \) is trivial on \( \Gamma(u) \setminus \{v\} \), so \( g \in G_u[1] \). Hence \( g \) fixes \( \Gamma_2(u) \cap \Gamma(v) \) setwise. By Lemma 2.2, \( g \) is not trivial on \( \Gamma_2(u) \). Hence \( \langle g \rangle \) has orbits of
size 3 on $\Gamma_2(u)$. Since $g$ fixes $\Gamma_2(u) \cap \Gamma(v_i)$ setwise and $|\Gamma_2(u) \cap \Gamma(v_i)| = 3$, it follows that $\langle g \rangle$ is transitive on $\Gamma_2(u) \cap \Gamma(v_i)$. Thus $G_{u,v_i}$ is transitive on $\Gamma_2(u) \cap \Gamma(v_i)$. Therefore $\Gamma$ is $(G, 2)$-geodesic-transitive. Then by [7, Corollary 1.4], $\Gamma$ is either the Octahedron or the Icosahedron. However, these two graphs do not have valency 6, which is a contradiction.

Finally, suppose that $|\Gamma(u) \cap \Gamma(w_i)| = 3$. Since there are 18 edges between $\Gamma(u)$ and $\Gamma_2(u)$, and $|\Gamma_2(u)| \cdot |\Gamma(u) \cap \Gamma(w_i)| = 18$, $|\Gamma_2(u)| = 6$. Since $|\Gamma_2(u) \cap \Gamma(w_i)| \geq 2$, $|\Gamma_3(u) \cap \Gamma(w_i)| \leq 1$. Thus by Remark 2.1, $\Gamma$ is $G$-distance-transitive. Inspecting the graphs in [3, p. 222-223], $\Gamma$ is the Paley graph $P(13)$. □

Lemma 2.8. Let $\Gamma$ be a $(G, 2)$-distance-transitive graph of valency 6. Let $u$ be a vertex of $\Gamma$. If $|\Gamma(u)| \equiv 2K_3$, then $|\Gamma_2(u)| = 9$ or 18.

Proof. Suppose that $|\Gamma(u)| \equiv 2K_3$. Then each arc lies in a unique $K_4$. Let $\Gamma(u) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ such that $(v_1, v_2, v_3)$ and $(v_4, v_5, v_6)$ are two triangles. Then for each $v_i$, $|\Gamma_2(u) \cap \Gamma(v_i)| = 3$. Since $|\Gamma(v_1)| \equiv 2K_3$, it follows that $\Gamma_2(u) \cap \Gamma(v_i) \cap \Gamma(v_j) = \emptyset$ for $i, j \in \{1, 2, 3\}$. Thus $|\Gamma_2(u)| \geq 9$.

On the other hand, since $\Gamma$ is $(G, 2)$-distance-transitive and $|\Gamma_2(u) \cap \Gamma(v_1)| = 3$, there are 18 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Thus $|\Gamma_2(u)|$ divides 18, and so $|\Gamma_2(u)| = 9$ or 18. □

If further $|\Gamma_2(u)| = 9$, then such a graph is unique.

Lemma 2.9. Let $\Gamma$ be a $(G, 2)$-distance-transitive graph of valency 6. Let $u$ be a vertex of $\Gamma$. Suppose that $|\Gamma(u)| \equiv 2K_3$ and $|\Gamma_2(u)| = 9$. Then $\Gamma \cong H(2, 4)$.

Proof. Since $|\Gamma(u)| \equiv 2K_3$, each arc lies in a unique $K_4$. Let $\Gamma(u) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Let $(v_1, v_2, v_3)$ and $(v_4, v_5, v_6)$ be the two triangles of $|\Gamma(u)|$. Then for each $v_i$, $|\Gamma_2(u) \cap \Gamma(v_i)| = 3$. Since $|\Gamma(v_1)| \equiv 2K_3$, it follows that $\Gamma_2(u) \cap \Gamma(v_i) \cap \Gamma(v_j) = \emptyset$ for $i \neq j \in \{1, 2, 3\}$. Since $|\Gamma_2(u)| = 9$, $\Gamma_2(u) = (\Gamma_2(u) \cap \Gamma(v_1)) \cup (\Gamma_2(u) \cap \Gamma(v_2)) \cup (\Gamma_2(u) \cap \Gamma(v_3))$. Set $\Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2, w_3\}$, $\Gamma_2(u) \cap \Gamma(v_2) = \{w_4, w_5, w_6\}$, and $\Gamma_2(u) \cap \Gamma(v_3) = \{w_7, w_8, w_9\}$. Since $|\Gamma(v_1)| \equiv |\Gamma(v_2)| \equiv |\Gamma(v_3)| \equiv 2K_3$, it follows that $(w_1, w_2, w_3)$, $(w_4, w_5, w_6)$ and $(w_7, w_8, w_9)$ are three triangles.

Since $\Gamma$ is $(G, 2)$-distance-transitive and $|\Gamma_2(u) \cap \Gamma(v_1)| = 3$, there are 18 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Since $|\Gamma_2(u)| = 9$, it follows that for each $w_i$, $|\Gamma(u) \cap \Gamma(w_i)| = 2$. By the previous argument, $w_1$ is not adjacent to any of $\{v_2, v_3\}$, so $w_1$ is adjacent to one of $\{v_4, v_5, v_6\}$, say $v_4$. Then $\Gamma(u) \cap \Gamma(w_1) = \{v_1, v_4\}$. As each arc lies in a unique $K_4$ and $(v_1, w_1, w_2, w_3)$ is a $K_4$, it follows that $v_4$ is not adjacent to any of $\{w_2, w_3\}$. Since $|\Gamma_2(u) \cap \Gamma(v_4)| = 3$ and $|\Gamma(v_1) \cap \Gamma(v_4)| = 2$ for $i = 1, 2, 3$, $v_4$ is adjacent to one of $\{w_4, w_5, w_6\}$, say $w_4$, and is adjacent to one of $\{w_7, w_8, w_9\}$, say $w_7$. Then $\Gamma_4(w_4) = \{u, v_5, v_6, w_1, w_4, w_7\}$. Since $|\Gamma(w_4)| \equiv 2K_3$ and $(u, v_5, v_6)$ is a triangle, it follows that $(w_1, w_4, w_7)$ is a triangle. Thus, $\Gamma(w_1) = \{v_1, v_4, w_2, w_3, w_4, w_7\}$, and so $\Gamma_3(u) \cap \Gamma(w_1) = \emptyset$. Since $\Gamma$ is $(G, 2)$-distance-transitive, it follows that $\Gamma$ is $G$-distance-transitive with diameter 2 and has 16 vertices. Thus by inspecting the graphs in [3, p. 222-223], $\Gamma \cong H(2, 4)$. □

Lemma 2.10. Let $\Gamma$ be a $(G, 2)$-distance-transitive graph of valency 6. Let $u$ be a vertex of $\Gamma$. If $|\Gamma(u)| \equiv 3K_2$, then $|\Gamma_2(u)| = 8, 12, or 24$.

Proof. Suppose that $|\Gamma(u)| \equiv 3K_2$. Then each arc lies in a unique triangle. Let $\Gamma(u) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ be such that $(v_1, v_2)$, $(v_3, v_4)$, and $(v_5, v_6)$ are three arcs. Then for
each $v_i$, $|\Gamma_2(u) \cap \Gamma(v_i)| = 4$. Since $|\Gamma(v_i)| \cong 3K_2$, it follows that $\Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_2) = \emptyset$. Thus $|\Gamma_2(u)| \geq 8$.

Since $\Gamma$ is $(G, 2)$-distance-transitive and $|\Gamma_2(u) \cap \Gamma(v_1)| = 4$, there are 24 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Since $|\Gamma_2(u)|$ divides 24, it follows that $|\Gamma_2(u)| = 8, 12, \text{ or } 24$. □

If further $|\Gamma_2(u)| = 8$, then $\Gamma$ is known.

**Lemma 2.11.** Let $\Gamma$ be a $(G, 2)$-distance-transitive graph of valency 6. Let $u$ be a vertex of $\Gamma$. Suppose that $|\Gamma(u)| \cong 3K_2$ and $|\Gamma_2(u)| = 8$. Then $\Gamma \cong KG_{6,2}$

**Proof.** Since $\Gamma$ is symmetric and $|\Gamma(u)| \cong 3K_2$, each arc lies in a unique triangle. Set $\Gamma(u) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Let $(v_1, v_2), (v_3, v_4)$ and $(v_5, v_6)$ be three arcs. Then for each $v_i$, $|\Gamma_2(u) \cap \Gamma(v_i)| = 4$. Since $|\Gamma(v_i)| \cong 3K_2$, it follows that $\Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_2) = \emptyset$. Since $|\Gamma_2(u)| = 8$, $\Gamma_2(u) = (\Gamma_2(u) \cap \Gamma(v_1)) \cup (\Gamma_2(u) \cap \Gamma(v_2))$. Set $\Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2, w_3, w_4\}$, and $\Gamma_2(u) \cap \Gamma(v_2) = \{w_5, w_6, w_7, w_8\}$. Since $|\Gamma_2(u)| \cong 3K_2$, it follows that $\Gamma_2(u)$ contains $\{w_1, w_2, w_3, w_4\}$ and $\{w_5, w_6\}$ and $(w_7, w_8)$ are arcs.

Since $\Gamma$ is $(G, 2)$-distance-transitive and $|\Gamma_2(u) \cap \Gamma(v_1)| = 4$, there are 24 edges between $\Gamma(u)$ and $\Gamma_2(u)$. As $|\Gamma_2(u)| = 8$, it follows that for each $w_i$, $|\Gamma(u) \cap \Gamma(w_i)| = 3$. By the previous argument, $w_1$ is not adjacent to $v_2$. Noting that $\Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_2) = \emptyset$ for $(i, j) = (1, 2), (3, 4), (5, 6)$, thus $w_1$ is adjacent to one of $\{v_3, v_4\}$, say $v_3$, and is also adjacent to one of $\{v_5, v_6\}$, say $v_5$. Then $\Gamma(u) \cap \Gamma(w_1) = \{v_1, v_3, v_5\}$. Since each arc lies in a unique triangle and $(v_1, w_1, w_2)$ is a triangle, it follows that $v_3$ is not adjacent to $w_2$. By $|\Gamma_2(u) \cap \Gamma(v_1)| = 4$ and $|\Gamma(v_1) \cap \Gamma(v_2)| = 3$ for $i = 1, 2, v_3$ is adjacent to one of $\{v_3, v_4\}$, say $v_3$, and is also adjacent to both vertices of $\{w_5, w_6, w_7, w_8\}$, say $w_5, w_7$.

Then $\Gamma(w_3) = \{u, v_4, w_1, w_2, w_3, w_7\}$. Since $|\Gamma(v_3)| \cong 3K_2$ and $(u, v_3)$ is an arc, it follows that $(w_1, w_5)$ and $(w_3, w_7)$ are two arcs. Thus, $\{v_1, v_3, v_5\} \cup \{w_2, w_5\} \subseteq \Gamma(u)$, and so $|\Gamma_2(u) \cap \Gamma(w_1)| \leq 1$. Since $\Gamma$ is $(G, 2)$-distance-transitive, it follows from Remark 2.1 that $\Gamma$ is $G$-distance-transitive. One part of the intersection array of $\Gamma$ is $(6, 4, \ldots; 1, 3, \ldots)$. By inspecting the graphs in [3, p.221], $\Gamma \cong KG_{6,2}$. □

**Lemma 2.12.** Let $\Gamma$ be an arc-transitive graph and let $u$ be a vertex of $\Gamma$. Suppose that $|\Gamma(u)| = U \cup W$, where $|U| = |W| = n$ and $U \cap W = \emptyset$. Assume further that $|U| \cong |W| \cong K_n$. Let $v_1 \in U$. If $|\Gamma(u) \cap \Gamma(v_1) \cap W| \leq n - 2$, then $\Gamma$ is a line graph.

**Proof.** Suppose that $|\Gamma(u) \cap \Gamma(v_1) \cap W| \leq n - 2$. Then $[U]$ and $[W]$ are the only two $n$-cliques of $\Gamma(u)$. It follows from [14, Proposition 2.1] that $\Gamma$ is a line graph. □

**Proof of Theorem 1.2.** Let $\Gamma$ be a connected non-complete $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive graph of valency 6. If $\Gamma$ has girth at least 5, then for any two vertices with distance 2, there exists a unique 2-arc between these two vertices. Thus $\Gamma$ is $(G, 2)$-arc-transitive, which is a contradiction. Hence $\Gamma$ has girth 3 or 4. If $\Gamma$ has girth 4, then it follows from Lemma 2.6 that $(\Gamma, G) = ((2 \times 7)-\text{grid}, S_2 \times M)$ where $M$ is a 2-transitive but not 3-transitive subgroup of $S_7$, so that (1) holds.

Suppose that $\Gamma$ has girth 3. Let $(u, v, w)$ be a 2-arc such that $d_\Gamma(u, w) = 2$. If $|\Gamma(u)|$ is connected, then by Lemma 2.7, $\Gamma$ is isomorphic to one of: $T(5)$, Paley graph $P(13)$, $K_3^{[3]}$ or $K_4^{[2]}$. (2) holds. If $|\Gamma(u)|$ is disconnected, then $G_u$ has blocks in $\Gamma(u)$, and each block has cardinality 2 or 3. If each block has cardinality 3, then $|\Gamma(u)| \cong 2K_3$; if each block has cardinality 2, then $|\Gamma(u)| \cong 2K_2$. Suppose that $|\Gamma(u)| \cong 2K_3$. Then by Lemma 2.8, $|\Gamma_2(u)| = 9 \text{ or } 18$. If $|\Gamma_2(u)| = 9$, then by Lemma 2.9, $\Gamma \cong \text{H}(2, 4)$. If $|\Gamma_2(u)| = 18$, then by Lemma 2.12, $\Gamma$ is a line graph, (3.1) holds.
Finally, if $[\Gamma(u)] \cong 3K_2$, then by Lemma 2.10, $|\Gamma_2(u)| = 8, 12, \text{ or } 24$. In particular, if $|\Gamma_2(u)| = 8$, then by Lemma 2.11, $\Gamma \cong KG_{6,2}$, so that (3.2) holds.

References


