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Comparing the irregularity and the total irregularity of graphs

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Abstract

Albertson [4] has defined the *irregularity* of a simple undirected graph G as $\operatorname{irr}(G) = \sum_{uv \in E(G)} |d_G(u) - d_G(v)|$, where $d_G(u)$ denotes the degree of a vertex $u \in V(G)$. Recently, in [1] a new measure of irregularity of a graph, so-called the *total irregularity*, was defined as $\operatorname{irr}_t(G) = \frac{1}{2} \sum_{u,v \in V(G)} |d_G(u) - d_G(v)|$. Here, we compare the irregularity and the total irregularity of graphs. For a connected graph G with n vertices, we show that $\operatorname{irr}_t(G) \leq n^2 \operatorname{irr}(G)/4$. Moreover, if G is a tree, then $\operatorname{irr}_t(G) \leq (n-2)\operatorname{irr}(G)$.

Keywords: The irregularity of graph, the total irregularity of graph.

Math. Subj. Class.: 05C05, 05C07,05C99

1 Introduction

Let G be a simple undirected graph of order $n = |V(G)|$ and size $m = |E(G)|$. For $v \in V(G)$, the degree of v, denoted by $d_G(v)$, is the number of edges incident to v. Albertson [4] defines the *imbalance* of an edge $e = uv \in E(G)$ as $\operatorname{imb}_G(uv) = |d_G(u) - d_G(v)|$ and the *irregularity* of G as

$$\operatorname{irr}(G) = \sum_{uv \in E(G)} \operatorname{imb}_G(uv). \tag{1.1}$$

Obviously, a connected graph G has irregularity zero if and only if G is regular. In [4] Albertson presented upper bounds on irregularity for bipartite graphs, triangle-free graphs

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and arbitrary graphs, as well as a sharp upper bound for trees. Some results about the irregularity of bipartite graphs are given in [4, 14]. Related to the work of Albertson is the work of Hansen and Mélot [13], who characterized the graphs with n vertices and m edges with maximal irregularity. Various upper bounds on the irregularity of a graph were given in [19], where K_{r+1}-free graphs, trees and unicyclic graphs with fixed number of vertices of degree one were considered. In [16], relations between the irregularity and the matching number of trees and unicyclic graphs were investigated. More results on irregularity, imbalance and related measures, one can find in [3, 5, 6, 17, 18].

Recently, in [1] a new measure of irregularity of a simple undirected graph, so-called the *total irregularity*, was defined as

$$\mathrm{irr}_t(G) = \frac{1}{2} \sum_{u,v \in V(G)} |d_G(u) - d_G(v)| \,. \tag{1.2}$$

Other approaches, that characterize how irregular a graph is, have been proposed [2, 3, 7, 8, 9, 10, 15]. In this paper, we focus on the relation between the irregularity (1.1) and the total irregularity (1.2) of a graph.

In the sequel we introduce the notation used in the rest of the paper. For $u, v \in V(G)$, we denote by $d_G(u, v)$ the length of a shortest path in G between u and v. In this short paper the notation of the sets, that will be defined next, is always regarding the graph G we consider. By $V_{a,b}$, we denote a set of vertices of a graph with degrees in $[a, b]$, and by $V_{\geq a}$ (resp. $V_{\leq a}$), we denote a set of vertices of a graph with degrees at least a (resp. with degrees at most a). Similarly, by $V^x_{\geq a}$ (resp. $V^x_{\leq a}$), we denote a set of neighboring vertices of a vertex x with degrees at least a (resp. with degrees at most a). The corresponding cardinalities of the above mentioned sets, we denote by small v (e.g., $v_{\leq a} = |V_{\leq a}|$ or $v^x_{\leq a} = |V^x_{\leq a}|$).

A subgraph $T = v_1 v_2 \cdots v_l$ of a graph G, where v_l is a leaf in G, is called a *tread* if $d_G(v_1) = d_G(v_2) = \cdots = d_G(v_{l-1}) = 2$, and v_1 is adjacent to a vertex with degree at least three. Let $T_1 = v_1 v_2 \cdots v_s$ and $T_2 = u_1 u_2 \cdots u_l$ be two threads of a graph G with leaves v_s and u_l, respectively, and let v_0 be the other neighbour of v_1. By $G' = G(T_2 \circ T_1)$ we denote a graph that is obtained from G after a *concatenation* of T_2 to T_1, i.e., after deleting the edge $v_0 v_1$ and adding an edge between u_l and v_1.

2 General graphs

Obviously, $\mathrm{irr}(G) \leq \mathrm{irr}_t(G)$. And, it is not hard to show that equality holds precisely when all non-adjacent vertices have same degree. Such a class of graphs are the complete k-partite graphs. More examples of graphs with equal irregularity and total irregularity can be found in [11]. Now, we give an upper bound on $\mathrm{irr}_t(G)$ in term of $\mathrm{irr}(G)$.

Theorem 2.1. Let G be a connected graph on n-vertices. Then

$$\mathrm{irr}_t(G) \leq \frac{n^2}{4} \mathrm{irr}(G).$$

Moreover, the bound is sharp for infinitely many graphs.

Proof. Let T be a spanning tree of G. Then, any two vertices a, b of G are connected by an unique path $P_{ab} = x_1 x_2 \cdots x_s$ in T, where $x_1 = a$ and $x_s = b$. By the triangle inequality, we have that

$$\begin{aligned} \mathrm{irr}_t(G) &= \frac{1}{2}\sum_{a,b\in V(G)} |d_G(a)-d_G(b)| \\ &\le \frac{1}{2}\sum_{a,b\in V(G)} |d_G(x_1)-d_G(x_2)|+|d_G(x_2)-d_G(x_3)|+\cdots \end{aligned} \tag{2.1}$$

For an edge $uv \in E(T)$, let $n_u = \{x \,|\, x \in V(T) \text{ and } d_T(x,u) < d_T(x,v)\}$. Similarly, let $n_v = \{x \,|\, x \in V(T) \text{ and } d_T(x,u) > d_T(x,v)\}$. Each summand $|d_G(u)-d_G(v)|$ in the last sum of (2.1) occurs in the sum exactly $n_{uv} = n_u n_v$ times. Also, each summand $|d_G(v)-d_G(u)|$ occurs n_{uv} times. Thus,

$$\mathrm{irr}_t(G) \le \sum_{uv\in E(T)} |d_G(u)-d_G(v)| n_{uv}.$$

As $n_{uv} \le (n/2)(n/2) = n^2/4$, and $\sum_{uv\in E(T)} |d_G(u)-d_G(v)| \le \sum_{uv\in E(G)} |d_G(u)-d_G(v)|$, we obtain the desired inequality.

Now, we show that the bound $n^2/4$ is sharp. Let a, b be two distinct integers, say $a < b$. Consider a graph G_a whose all vertices are of degree a, with exception of one vertex u which is of degree $a-1$. Similarly, consider a graph G_b whose all vertices are of degree b, with exception of one vertex u which is of degree $b-1$. Let G^* be the graph obtained from G_a and G_b by connecting u and v. Let $n_a = |V(G_a)|$ and $n_b = |V(G_b)|$. Observe that $\mathrm{irr}(G^*) = b-a$ and $\mathrm{irr}_t(G^*) = (b-a)n_a n_b$. Choosing $n_a = n_b = n/2$, we obtain

$$\frac{\mathrm{irr}_t(G^*)}{\mathrm{irr}(G^*)} = n_a n_b = \frac{n^2}{4}.$$

In order to show that such graphs G_a and G_b exist, one may use the theorem of Erdős-Gallai [12] which states that a sequence $d_1 \ge d_2 \ge \cdots \ge d_n$ of non-negative integers with even sum is graphic (i.e., there exist a graph with such a degree sequence) if and only if

$$\sum_{i=1}^{r} d_i \le r(r-1) + \sum_{i=r+1}^{n} \min(r, d_i), \tag{2.2}$$

for all $1 \le r \le n$.

So, fix a, b, and $n_a = n_b$ to be odd numbers with $n_a \gg \max\{a, b\}$. We will show the existence of the graph G_a. In a similar way, one can show the existence of the graph G_b. As $(n_a-1)a + (a-1)$ is even, the parity condition of the theorem of Erdős-Gallai is satisfied. So, we need to show only (2.2). For this we consider three cases regarding r and a:

- $r \le a-1$. Then, (2.2) can be written as $ra \le r(r-1) + (n_a - r)r$. It obviously holds since $a \ll n_a - r$.
- $r = a$. In this case, (2.2) can be written as $ra \le r(r-1) + (n_a - r)r - 1$, which holds for a similar reason as the previous case.
- $r \ge a+1$. Similarly, (2.2) can be written as $ra \le r(r-1) + (n_a - r)a - 1$, and it holds as $ra \ll r(r-1)$.

□

3 Trees

In this section, we give an upper bound on $\operatorname{irr}_t(G)$ in term of $\operatorname{irr}(G)$, when G is a tree. To show the bound, we will use the following lemma.

Lemma 3.1. *Let G be a tree, x a vertex of degree $d \geq 3$ incident with threads T_1 and T_2, and let $G' = G(T_2 \circ T_1)$. Then,*

(a) $\operatorname{irr}_t(G) - \operatorname{irr}_t(G') = 2v_{2,d-1}$*;*

(b) $\operatorname{irr}(G) - \operatorname{irr}(G') = 2(d - v^x_{\geq d} - 1)$.

Proof. Let $T_1 = a_1 a_2 \cdots a_s$ and $T_2 = b_1 b_2 \cdots b_l$. We consider the identities separately.

(a) Notice that all other vertices except x and b_l have the same degree in G and G'. Hence, it holds that

$$\begin{aligned}\operatorname{irr}_t(G) - \operatorname{irr}_t(G') &= \sum_{u \neq b_l} (|d_G(x) - d_G(u)| - |d_{G'}(x) - d_{G'}(u)|) \\ &\quad + \sum_{u \neq x} (|d_G(u) - d_G(b_l)| - |d_{G'}(u) - d_{G'}(b_l)|) \\ &\quad + |d_G(x) - d_G(b_l)| - |d_{G'}(x) - d_{G'}(b_l)|.\end{aligned}$$

Since $d_{G'}(x) = d_G(x) - 1 = d - 1$ and $d_{G'}(b_l) = d_G(b_l) + 1 = 2$, further we have

$$\begin{aligned}\operatorname{irr}_t(G) - \operatorname{irr}_t(G') &= \sum_{u \neq b_l} (|d - d_G(u)| - |d - 1 - d_G(u)|) \\ &\quad + \sum_{u \neq x} (|d_G(u) - 1| - |d_G(u) - 2|) + 2. \qquad (3.1)\end{aligned}$$

If $u \in V_{\leq d-1}$, then $|d - d_G(u)| - |d - 1 - d_G(u)| = 1$, otherwise $|d - d_G(u)| - |d - 1 - d_G(u)| = -1$. Hence, the first sum in (3.1) is equal to $v_{\leq d-1} - 1 - v_{\geq d}$. Similarly, if $u \in V_{\geq 2}$, then $|d_G(u) - 1| - |d_G(u) - 2| = 1$, otherwise $|d_G(u) - 1| - |d_G(u) - 2| = -1$. Thus, the second sum in (3.1) is equal to $v_{\geq 2} - 1 - v_1$. Applying these observations, we have

$$\begin{aligned}\operatorname{irr}_t(G) - \operatorname{irr}_t(G') &= v_{\leq d-1} - 1 - v_{\geq d} + v_{\geq 2} - 1 - v_1 + 2 \\ &= v_{\leq d-1} - v_1 + v_{\geq 2} - v_{\geq d} \\ &= 2v_{2,d-1}.\end{aligned}$$

(b) Let $e_1 = xa_1$, $e_2 = xb_1$, $e_3 = b_{l-1}b_l$ and $E_1 = \{e_1, e_2, e_3\}$. Denote by E_2 the set of edges incident to x that are different from e_1 and e_2. Notice that every edge not in $E_1 \cup E_2$ contributes zero to the difference $\operatorname{irr}(G) - \operatorname{irr}(G')$. So, we can infer

$$\begin{aligned}\operatorname{irr}(G) - \operatorname{irr}(G') &= \sum_{uv \in E_2} (\operatorname{imb}_G(uv) - \operatorname{imb}_{G'}(uv)) \\ &\quad + \sum_{uv \in E_1} (\operatorname{imb}_G(uv) - \operatorname{imb}_{G'}(uv)).\end{aligned}$$

Notice that the first sum is equal to $-v^x_{\geq d} + (v^x_{\leq d-1} - 2)$ (we have -2 as the edges e_1 and e_2 are excluded in this sum). In G', the edge $e_1 = xa_1$ does not exist anymore, but there is a new edge $e'_1 = b_l a_1$. Observe that after the concatenation $T_2 \circ T_1$ all other edges preserve their end-vertices. First, we consider the contribution of e_1 and e'_1 in $\mathrm{irr}(G) - \mathrm{irr}(G')$. There are two possibilities regarding the length of T_1:

- $s = 1$: Then, $\mathrm{imb}_G(e_1) = d - 1$ and $\mathrm{imb}_{G'}(e'_1) = 1$;
- $s \geq 2$: In this case, $\mathrm{imb}_G(e_1) = d - 2$ and $\mathrm{imb}_{G'}(e'_1) = 0$.

In both of them, we obtain $\mathrm{imb}_G(e_1) - \mathrm{imb}_{G'}(e'_1) = d - 2$.

Next, we consider the contributions of e_2 and e_3 together. Again, consider two possibilities regarding the length of T_2:

- $l = 1$: Then, $e_2 = e_3$ and $\mathrm{imb}_G(e_2) = d - 1$ and $\mathrm{imb}_{G'}(e_2) = d - 3$;
- $l \geq 2$: In this case, $e_2 \neq e_3$, and $\mathrm{imb}_G(e_2) = d - 2$, $\mathrm{imb}_{G'}(e_2) = d - 3$, $\mathrm{imb}_G(e_3) = 1$ and $\mathrm{imb}_{G'}(e_3) = 0$.

In both cases, we obtain that $\sum_{e \in \{e_2, e_3\}} (\mathrm{imb}_G(e) - \mathrm{imb}_{G'}(e)) = 2$. So finally, we have that

$$\begin{aligned}
\mathrm{irr}(G) - \mathrm{irr}(G') &= -v^x_{\geq d} + (v^x_{\leq d-1} - 2) + d - 2 + 2 \\
&= -v^x_{\geq d} + v^x_{\leq d-1} - 2 + d \\
&= 2(d - v^x_{\geq d} - 1).
\end{aligned}$$

□

Theorem 3.1. Let G be a tree with n vertices. Then

$$\mathrm{irr}_t(G) \leq (n-2)\mathrm{irr}(G).$$

Moroever, equality holds if and only if G is a path.

Proof. Let $n_1(G)$ be the number of vertices of G with degree one. We will prove the second inequality by induction on $n_1(G)$. If $n_1(G) = 0$, then $G \simeq P_1$, $\mathrm{irr}(G) = \mathrm{irr}_t(G) = 0$, and the equality in the theorem holds. Since G is a tree, $n_1(G) \neq 1$. If $n_1(G) = 2$, then $G \simeq P_n$. In this case $\mathrm{irr}(G) = 2$ and $\mathrm{irr}_t(G) = 2(n-2)$, hence we obtain equality.

Now, assume $n_1(G) > 2$. Then, it is easy to see that G has a vertex x of degree $d \geq 3$, incident with at least two threads T_1 and T_2. Let $G' = G(T_2 \circ T_1)$. Since $n_1(G') = n_1(G) - 1$, we can assume that inequality holds for G', i.e.,

$$\mathrm{irr}_t(G') \leq (n-2)\mathrm{irr}(G'). \tag{3.2}$$

By Lemma 3.1, we have

$$\mathrm{irr}(G') = \mathrm{irr}(G) - 2(d - v^x_{\geq d} - 1) \qquad \text{and} \qquad \mathrm{irr}_t(G') = \mathrm{irr}_t(G) - 2v_{2,d-1}. \tag{3.3}$$

Plugging (3.3) in (3.2), we obtain

$$(n-2)\mathrm{irr}(G) \geq \mathrm{irr}_t(G) - 2v_{2,d-1} + 2(n-2)(d - v^x_{\geq d} - 1). \tag{3.4}$$

As $d(x) = d \geq 3$ and x is incident with two threads, we infer $v^x_{\geq d} + 2 \leq d$, and so $2(d - v^x_{\geq d} - 1) \geq 2$. Observe also that $v_{2,d-1} \leq n - 3$. Hence $2(n-2)(d - v^x_{\geq d} - 1) > 2(n-3) \geq 2v_{2,d-1}$. This together with (3.4) gives $(n-2)\mathrm{irr}(G) > \mathrm{irr}_t(G)$. □

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