

On list-coloring extendable outerplanar graphs*

Joan P. Hutchinson

Macalester College, St. Paul, MN 55105, USA

Received 7 November 2010, accepted 7 September 2011, published online 1 February 2012

Abstract

We investigate a variation on Thomassen's 2- and 3-extendability of precoloring extensions for list-coloring graphs. For an outerplanar graph G with $i, j \leq 2$, we say that G is $\{i, j\}$ -extendable if for every pair of nonadjacent vertices x and y , whenever x is assigned an i -list, y is assigned a j -list, and all other vertices have a 3-list, G is list-colorable. We characterize the $\{1, 1\}$ - and the $\{1, 2\}$ -extendable outerplanar graphs and prove that every outerplanar graph is $\{2, 2\}$ -extendable.

Keywords: Graph list-coloring, coloring extendability, outerplanar graphs.

Math. Subj. Class.: 05C15

1 Introduction

In C. Thomassen's celebrated theorem that every planar graph is 5-list-colorable [7] (see also [1]), he proves a stronger result.

Theorem 1.1. *If G is a plane near-triangulation with outer cycle C , if two consecutive vertices of C are precolored (with different colors), if all other vertices of C have a 3-list, and if every other vertex of G has a 5-list, then the precoloring extends to a list-coloring of G .*

In [8] he calls these conditions the property of being 2-extendable, and characterizes the planar graphs that are 3-extendable. That is, if G is a plane near-triangulation with outer cycle C , if three consecutive vertices of C , v_1, v_2, v_3 , are properly precolored, if every other vertex on C has a 3-list, and if every other vertex of G has a 5-list, then the graph is called 3-extendable with respect to the path v_1, v_2, v_3 if it is list-colorable. Not all plane graphs are 3-extendable with respect to every 3-path on the outer cycle.

We consider an analogous list-coloring problem when, in a plane graph with outer facial walk W , two nonadjacent vertices are precolored, the remaining vertices of W have

*In memory of Michael O. Albertson, 1946–2009.

E-mail address: hutchinson@macalester.edu (Joan P. Hutchinson)

3-lists, and all other vertices have 5-lists. When the graph is list-colorable, we say it is $\{1, 1\}$ -extendable. Not all plane graphs are $\{1, 1\}$ -extendable. More generally we consider the cases when two nonadjacent vertices have lists of size i and j , with $i, j = 1$ or 2 , and call these $\{i, j\}$ -extendable when there is a list-coloring. In this paper we consider *outerplanar* graphs with nonadjacent vertices x and y , with x having an i -list, y a j -list, and all other vertices having a 3-list. We characterize the $\{1, 1\}$ -extendable and $\{1, 2\}$ -extendable outerplanar graphs and prove that every outerplanar graph is $\{2, 2\}$ -extendable. We formulate questions about possible generalizations of these results to all planar graphs.

Albertson [2] has asked whether there is a distance $d > 0$ such that if vertices of a set P , mutually at distance at least d in a planar G , are precolored, then the precoloring extends to a 5-list-coloring of G ; Tuza and Voigt [9] have shown that in general distance 4 is not enough. In [3] we prove that the answer is yes when P consists of vertices on a common face and mutually at distance at least three. Thus these graphs are $\{1, 1\}$ -“5-list-extendable,” meaning that all vertices except for P must have 5-lists. In contrast suppose G is a plane near-triangulation with outer cycle C containing two nonadjacent precolored vertices, with all other vertices of C having a 4-list, and with all other vertices having a 5-list. Then in Section 6, using the techniques of [3], we show that with these sizes of lists on vertices of C all planar graphs are $\{1, 1\}$ -“4-list-extendable.”

2 Background

We follow the terminology of [6, 7, 10]. Let G be a graph. When G is planar and embedded in the plane, we call it a *plane* graph. For every $v \in V(G)$, let $L(v)$ be a list of colors assigned to v . When $|L(v)| = 1$, we may say that v is precolored with its one available color. G is said to be L -list-colorable (or when L is clear, *list-colorable*) when the vertices can be properly colored so that each vertex v receives a color from $L(v)$. As in [7], it is natural to consider near-triangulations for this coloring problem. A plane graph becomes a (plane) near-triangulation by the addition of edges so that every interior, finite face becomes bounded by three edges; such a face is called a *triangle*. The addition of these edges only makes the coloring problem more difficult, yet we remain in the class of plane graphs.

An *outerplanar* graph is one that has a plane embedding in which all vertices lie on the exterior, infinite face; when it is so embedded, it is called an *outerplane* graph. Every outerplane graph can be extended to an edge-maximal, 2-connected, near-triangulation that remains outerplane. Then all vertices lie on a cycle that bounds the exterior face, and all interior faces are triangles. We shall also consider 1-connected, outerplane near-triangulations in which all vertices lie on a walk that bounds the exterior face, and all interior faces are triangles. Associated with such a 1-connected, outerplane near-triangulation is the block-cutpoint tree; see [10].

Here is an immediate consequence of Thm. 1.1.

Corollary 2.1. *Let G be a 1-connected near-triangulation with outer boundary W , and let $x, y \in V(W)$. If x and y are adjacent and have, respectively, an i -list and a j -list, $i, j = 1, 2$, or 3 , if all vertices of $W \setminus \{x, y\}$ have a 3-list, and all vertices of $V(G) \setminus V(W)$ have a 5-list, then G is list-colorable unless x and y have the same 1-list.*

Proof. The vertices x and y can always be properly (pre) list-colored under these hypotheses. The result follows from Thm. 1.1 when G is 2-connected. If G has cut-vertices, then since the block B containing x and y can be list-colored, that list-coloring extends to the rest of G by Thm. 1.1. \square

Because of Cor. 2.1 we consider list-coloring extensions of two nonadjacent vertices. First we consider the problem for 2-connected, outerplane graphs and then expand the result to outerplane graphs with cut-vertices and in which the nonadjacent vertices lie in different blocks. We note that Cor. 2.1 applied to outerplane near-triangulations can be proved easily and directly without reference to Thm. 1.1.

Definition 2.2. Let G be a 2-connected near-triangulation with outer cycle C and nonadjacent vertices $x, y \in V(C)$. Let $i, j \in \{1, 2\}$.

1. G is said to be $\{i, j\}$ -extendable with respect to (nonadjacent) vertices x, y , if whenever x is assigned an i -list $L(x)$, y is assigned a j -list $L(y)$, every vertex $v \in V(C) \setminus \{x, y\}$ is assigned a 3-list $L(v)$, and every vertex $w \in V(G) \setminus V(C)$ is assigned a 5-list $L(w)$, then G is L -list-colorable.
2. G is said to be $\{i, j\}$ -extendable if for every choice of nonadjacent $x, y \in V(C)$, it is $\{i, j\}$ -extendable with respect to vertices x and y .

In [8] Thomassen characterizes the 3-extendable near-triangulations with respect to a 3-path of vertices on the outer cycle in terms of "broken wheels" and "generalized wheels." Needed for our study of outerplane graphs, a broken wheel is the near-triangulation that consists of a k -cycle on v_1, v_2, \dots, v_k plus all diagonals from v_1 : $v_1v_3, v_1v_4, \dots, v_1v_{k-1}$. (This is a k -wheel minus the edge v_2v_k .)

Here is a corollary that follows immediately from the 3-extendability theorem and proof of [8].

Corollary 2.3. Let G be a 2-connected, outerplane near-triangulation with outer cycle $C = x_1, x_2, \dots, x_n$. Reading subscripts modulo n ,

1. for $i = 1, 2, \dots, n$, G is 3-extendable with respect to the 3-path x_i, x_{i+1}, x_{i+2} if and only if x_i is adjacent to x_{i+2} , and
2. for $i = 1, 2, \dots, n$, G is not $\{1, 1\}$ -extendable with respect to x_i, x_{i+2} (assuming that these vertices are not adjacent).

Proof. (1.) If x_i and x_{i+2} are adjacent, then a proper coloring of x_i, x_{i+1}, x_{i+2} extends to G by Thm. 1.1.

(1. and 2.) If x_i and x_{i+2} are not adjacent, when $L(x_i) = \{a\}$, $L(x_{i+2}) = \{c\}$ with $a \neq c$, and $L(x_{i+1}) = \{a, b, c\}$, then there is exactly one list-coloring of the 3-path x_i, x_{i+1}, x_{i+2} , and by [8] this does not extend since G must contain a broken wheel at x_i, x_{i+1}, x_{i+2} . □

Similarly one can check that for $i = 1, 2, \dots, n$, G is not $\{1, 2\}$ -extendable with respect to x_i, x_{i+2} (when x_i and x_{i+2} are not adjacent) if and only if the degree of x_{i+1} is odd. G is always $\{2, 2\}$ -extendable with respect to x_i, x_{i+2} . These results will follow also from those of Theorem 3.3.

3 $\{i, j\}$ -extendability of 2-connected, outerplane near-triangulations, $i, j \leq 2$

For later work, we include all 2-connected, outerplane graphs in the beginning of this section.

Definition 3.1. Let x and y be nonadjacent vertices of a 2-connected, outerplane graph G . Let T be the weak dual of G with each vertex of T representing an interior region of G . Let $p_{x,y}$ represent the shortest path in T from a region incident with x to a region incident with y , and let the *fundamental $x - y$ subgraph* of G , $G_{x,y}$, consist of the subgraph of G on these same regions, their edges and vertices.

An example is shown in Figure 1(a) when G is a near-triangulation, and in 1(c) when G is not a near-triangulation. The corresponding $G_{x,y}$ are illustrated in (b) and (d). In the former case $G_{x,y}$ is also a 2-connected, outerplane near-triangulation in which x and y are two, and the only two, vertices of degree 2; see Figure 1(b). In the latter case, edges can be added to interior faces of $G_{x,y}$ so that x and y are precisely the two vertices of degree 2; see Figure 1(d).

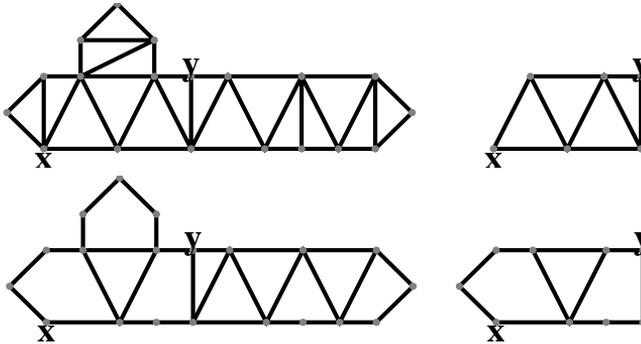


Figure 1: (a) An outerplanar near-triangulation G with selected vertices x and y , (b) $G_{x,y}$, (c) a non-near-triangulation G' with selected vertices, x and y , (d) $G'_{x,y}$.

Proposition 3.2. Let x and y be nonadjacent vertices of a 2-connected, outerplane graph G with fundamental $x - y$ subgraph $G_{x,y}$. If $|L(v)| \leq 2$ for $v = x, y$, and $|L(v)| = 3$ for all $v \in V(G) \setminus \{x, y\}$, then G can be list-colored if and only if $G_{x,y}$ can be.

Proof. If $G_{x,y}$ can be list-colored, then this extends to a list-coloring of G by Thm. 1.1. \square

In the rest of this section we focus on 2-connected, outerplane near-triangulations in which there are precisely two degree-2 vertices and these two are the vertices with small lists; in a near-triangulation, these two cannot be adjacent. The essence of the situation is illustrated in Figure 2.

Notice that 2-connected, outerplane near-triangulations are uniquely 3-colorable so that we may talk about the 3-color class of x , CC_x . Thus in Figure 2(a), x and y lie in the same color class, CC_x , and in Figure 2b, $y \notin CC_x$. Here is the goal of this section.

Theorem 3.3. Let G be a 2-connected, outerplane near-triangulation with exactly two degree-2 vertices x and y .

1. G is not $\{1, 1\}$ -extendable with respect to x and y .
2. G is not $\{1, 2\}$ -extendable with respect to x and y if and only if $y \in CC_x$.
3. G is $\{2, 2\}$ -extendable with respect to x and y .

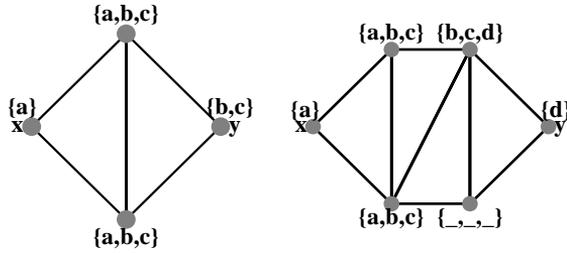


Figure 2: (a) A non- $\{1, 2\}$ -extendable graph, (b) a non- $\{1, 1\}$ -extendable graph.

In addition we will characterize the lists that make each G non-extendable in (1) and (2). Thm. 3.3 has also been proved in a different manner by Axenovich and Lastrina [4] as they explore additional conditions that ensure that a graph is $\{2, 2\}$ -extendable with respect to two vertices.

Let G be a 2-connected, outerplane near-triangulation with exactly two degree-2 vertices x and y . We label the vertices of G as follows: let $x = v_0$ and label its neighbors v_1 and v_2 . Without loss of generality v_1 has degree 3 in G ; label its third neighbor v_3 . Then the triangle $\{v_1, v_2, v_3\}$ lies on one side of the edge v_2v_3 , and the third (unlabeled) vertex of the triangle lying on the other side of v_2v_3 is labeled v_4 . We continue labeling vertices by the succession of triangles, each incident with the preceding. In the final triangle containing y , the neighbors of y will have been labeled v_j and v_{n-2} for some $j < n - 2$, and y is labeled $y = v_{n-1}$. For $i = 2, 3, \dots, n - 1$, let G_i denote the induced subgraph of G on vertices v_0, v_1, \dots, v_i . For $i = 2, \dots, n - 1$, each triangle consists of $\{v_{t_i}, v_{i-1}, v_i\}$ for some $t_i < i - 1$, and we call v_{i-1} and v_{t_i} the predecessors of v_i . An example is shown in Figure 3(a).

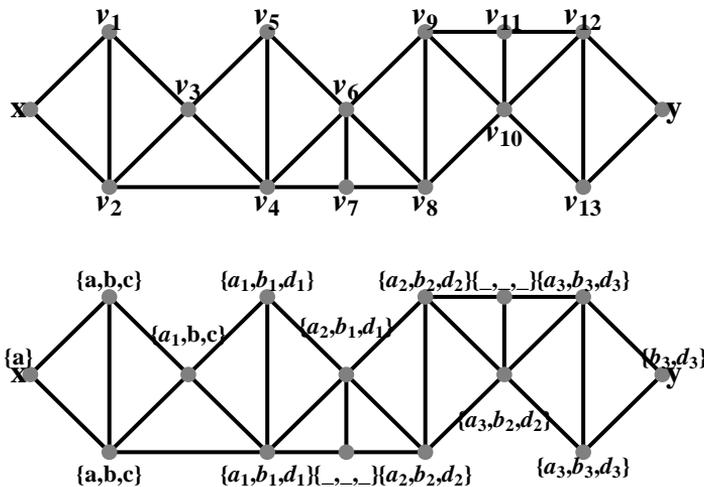


Figure 3: (a) A labeled, 2-connected, outerplane near-triangulation, (b) the graph with a set of noncoloring lists.

We note some additional properties of the near-triangulation G : Every triangle of G contains precisely one vertex of CC_x , the color class of x in a 3-coloring. With a precoloring of x , the remaining graph is 3-list-colorable by Thm. 1.1, though maybe not when y has a small list. The *sequential list-coloring algorithm* of $G \setminus \{y\}$ begins by extending the coloring of x to its two neighbors and then in general extends to the next-indexed vertex, adjacent to two colored neighbors. With 3-lists there is always a color available for each extension in $G \setminus \{y\}$.

We enumerate and count the number of possible list-colorings of vertex v_i in G_i , using variables $C(i)$ and $m(i) = |C(i)|$; we may say that $C(i)$ is the C -set and $m(i)$ the m -value of v_i . Suppose $|L(x)| = 1$ and $|L(v)| = 3$ for $v \in V(G) \setminus \{x, y\}$. For $i = 0, 1, \dots, n - 2$ let $C(i) \subseteq L(v_i)$ be defined by: $C(0) = L(v_0)$ and $C(1) = L(v_1) \setminus C(0)$. For $i \geq 2$, v_i lies on a triangle formed with its two predecessors v_{i-1} and v_{t_i} for some $t_i < i - 1$. We define $C(i) \subseteq L(v_i)$ to be the list of colors that are possible on v_i , given the lists of possible colors $C(i - 1)$ and $C(t_i)$. In other words $C(i) \subseteq L(v_i)$ contains the color a if and only if there is a list-coloring of G_i in which v_i receives color a . As we've noted, $m(i) \geq 1$ for all $i \leq n - 2$.

For example when $|L(x)| = 1$, for $i = 1, 2$ $C(i) = L(v_i) \setminus C(0)$ so that $m(i) = 2$ or 3 . As seen in Figure 2 with $C(0) = \{a\}$, $m(1) = 2$ when $L(v_1) = \{a, b, c\}$, and $m(2) = 2$ when $L(v_2) = \{a, b', c'\}$. In these cases $C(0)$ is disjoint from $C(1)$ and from $C(2)$ with $v_1, v_2 \notin CC_x$. Continuing with this example, $m(3) = 1$ if and only if $L(v_1) = L(v_2) = \{a, b, c\}$ and $L(v_3) = \{b, c, d\}$ where d may or may not equal a , but not b or c . In this case $C(1) = C(2)$, $m(1) = m(2) = 2$, $C(3) = \{d\}$, $|C(1) \cap L(v_3)| = 2$, and $C(3)$ is disjoint from $C(1)$ and $C(2)$. Similarly $m(3) = 2$ if and only if $L(v_1) = L(v_2) = \{a, b, c\}$ and $L(v_3) = \{b, d, e\}$ or $\{c, d, e\}$ where d or e may equal a . In this case again $C(1) = C(2)$, $m(1) = m(2) = 2$, $|C(1) \cap L(v_3)| = 1$, and $C(3) = \{d, e\}$ is disjoint from $C(1)$ and $C(2)$. One can easily check that when $C(1) \neq C(2)$ or when $m(1)$ or $m(2) = 3$, then $C(3) = L(v_3)$ and $m(3) = 3$.

In most cases $m(i) = 3$, but lists that prevent list-coloring are constrained ones on the vertices of CC_x and their predecessors. Specifically we will characterize when for every $v_i \in CC_x$, $m(i) = 1$, and this characterization will lead to lists that prevent list-coloring.

Lemma 3.4. *Let G be a 2-connected, outerplane near-triangulation with exactly two degree-2 vertices, with notation as above, and with all vertices having 3-lists except that $|L(x)| = 1$ and $|L(y)| \leq 2$. For $i > 0$,*

1. *when $v_i \in CC_x$, $m(i) < 3$ if and only if v_i 's predecessors have $C(i - 1) = C(t_i)$, $m(i - 1) = 2$, and $C(i - 1) \cap L(v_i) \neq \emptyset$,*
2. *when $v_i \in CC_x$ and $m(i) < 3$ and v_j is a predecessor of v_i , then $C(j)$ and $C(i)$ are disjoint,*
3. *when $v_i \notin CC_x$, $m(i) \geq 2$, and*
4. *if $v_i v_j$ is an edge with $j < i$, $C(i) = C(j)$, and $m(i) = 2$, then v_i and v_j are adjacent to v_k in CC_x , $k < i$, with $m(k) = 1$.*

Proof. The proof is by induction on i , and the base cases were presented in a preceding paragraph. Let $i > 3$ and $v_i \in CC_x$. If the conditions of 1 hold, then $m(i) < 3$. If $m(i) < 3$, by induction v_{t_i} and v_{i-1} satisfy condition 3. Since $m(t_i), m(i - 1) \geq 2$ and $m(i) < 3$, then we must have $m(t_i) = m(i - 1) = 2$ with $C(t_i) = C(i - 1)$ implying that $m(i) = 1$ or 2 depending on whether $|C(i - 1) \cap L(v_i)| = 2$ or 1 , respectively. In addition

let $j = i - 1$ or t_i so that v_j is a predecessor of v_i . Then since $C(i - 1) \cap L(v_i) \neq \emptyset$, $C(i - 1) = C(t_i)$ is disjoint from $C(i)$ and condition 2 holds.

Let $v_i v_j, j < i$, be an edge with $C(i) = C(j)$ and $m(i) = 2$ so that v_j is a predecessor of v_i . If the other predecessor of v_i is $v_{j'}$ with $j' < i$, then if $m(j') = 3$, also $m(i) = 3$, a contradiction. If $m(j') = 1$, then $v_{j'} \in CC_x$ by induction and condition 3 so that condition 4 holds. Otherwise $m(j') = 2$. Note that if $C(j) \neq C(j')$, then $m(i) = 3$. Thus $C(j') = C(j) = C(i)$, but that is impossible, and we conclude that condition 4 holds.

If $v_i \notin CC_x$, then one predecessor of v_i , say v_j , lies in CC_x and the other, $v_{j'}$, is not in CC_x . By induction $m(j') \geq 2$, and if $m(j) = 3$, then $m(i) = 3$, and if $m(j') = 3$, then $m(i) \geq 2$ as claimed. Thus assume $m(j') = 2$ and $m(j) \leq 2$. If $j' < j$, then $v_{j'}$ is a predecessor of v_j and by condition 2, $C(j)$ and $C(j')$ are disjoint implying $m(i) \geq 2$. If $j < j'$ and $m(j) = 1$, then necessarily $C(j)$ and $C(j')$ are disjoint, implying $m(i) \geq 2$. If $j < j'$ and $m(j) = 2$, then by induction and condition 4, $C(j) \neq C(j')$, implying that $m(i) \geq 2$. In all cases condition 3 holds. \square

Corollary 3.5. *Let G satisfy the hypotheses of Lemma 3.4.*

1. *If $v_i v_j$ is an edge with $m(i) = 1$ and $m(j) = 2$, then $C(i)$ and $C(j)$ are disjoint.*
2. *If $v_i v_j$ is an edge with $j < i$, $C(i) = C(j)$, and $m(i) = 2$, then $v_i, v_j \notin CC_x$.*
3. *If $v_i v_j$ is an edge, $v_i \in CC_x$, $C(i) \neq C(j)$ but $m(i) = m(j) = 2$, then $C(i)$ and $C(j)$ are disjoint.*

Proof. 1. If $i < j$, then this is immediate. If $j < i$, then v_j is a predecessor of v_i , and by Lemma 3.4(2) the sets are disjoint.

2. This follows from Lemma 3.4(4).

3. If $j < i$, this follows from Lemma 3.4(2). If $i < j$, let $v_{i'}$ be the other predecessor of v_j . By Lemma 3.4(3), $m(i') \geq 2$. If $m(i') = 3$, then $m(j) = 3$, a contradiction. Thus $m(i') = 2$. If $C(i) \neq C(i')$, then $m(j) = 3$, a contradiction. Thus $C(i) = C(i')$ and since $m(j) = 2$, $C(i)$ and $C(j)$ must be disjoint. \square

Proposition 3.6. *Let G be a 2-connected, outerplane near-triangulation with exactly two degree-2 vertices and with notation as above. Suppose $|L(x)| = 1$ and $|L(y)| \leq 2$. Then G is not list-colorable if and only if for each $v_i \in CC_x$, $m(i) = 1$ and*

1. *if $y = v_{n-1} \in CC_x$, then its predecessors have $C(n-2) = C(t_{n-1})$, $m(n-2) = 2$, and $L(y) \subseteq C(n-2)$, or*
2. *if $y \notin CC_x$, then $|L(y)| = 1$ and for its predecessor $v_i \in CC_x$, $C(i) = L(y)$.*

Proof. If $m(i) = 1$ for every $v_i \in CC_x$ and conditions (1) or (2) hold, then G is not list-colorable.

Conversely suppose G is not list-colorable; we know that $G \setminus \{y\}$ is list-colorable. If $y \in CC_x$, then its two predecessors do not lie in CC_x and by Lemma 3.4(3) have m -values at least 2. Since G is not list-colorable, they must have equal C -sets of size 2 with $L(y)$ a subset of these sets. If $y \notin CC_x$, then one predecessor, say v_i , lies in CC_x , the other, v_j , does not and $m(j) \geq 2$. If $m(i) = 3$, then G would be list-colorable implying that $m(i) < 3$. If $m(j) = 3$, then G is not list-colorable only when the conditions of 2 hold. Similarly if $m(i) = 1$ and $m(j) = 2$, then by Cor.3.5(1) $C(i)$ and $C(j)$ are disjoint and the

conditions of 2 hold. Thus suppose $m(i) = 2$ and $m(j) = 2$. By Cor.3.5(2), $C(i) \neq C(j)$ and by Cor.3.5(3) $C(i)$ and $C(j)$ are disjoint. Thus G is list-colorable, a contradiction. We conclude that conditions 1 or 2 must hold. \square

In summary, for every 2-connected, outerplane near-triangulation with exactly two degree-2 vertices, there is a set of 3-lists such that when the degree-2 vertices are pre-colored (or one has a choice of two colors), the graph is not list-colorable. We call such lists *noncoloring lists*.

As the examples of Figures 2(b) and 3(b) show, there may be some lists within such a non-list-colorable graph that are not constrained, but lists are constrained on an important subgraph of G , a *chain of diamonds*, which we shall use again in Section 4. We relabel the vertices of this chain by letting $x_0 = x, x_3, x_6, \dots, x_{3k}$ be the vertices of CC_x listed with increasing index, and suppose that each has m -value 1. Then by Lemma 3.4(1), for $i = 1, \dots, k$ each x_{3i} has two predecessors, x_{3i-1} and x_{3i-2} , with identical C -sets and m -value 2, and by 3.4(4) these two have a naturally associated $x_{3i-3} \in CC_x$. The subgraph on these vertices and y forms the chain of diamonds and appears as in Figure 4 where we illustrate the cases when $y \in CC_x$ so that $y = x_{3k}$, and when $y \notin CC_x$.

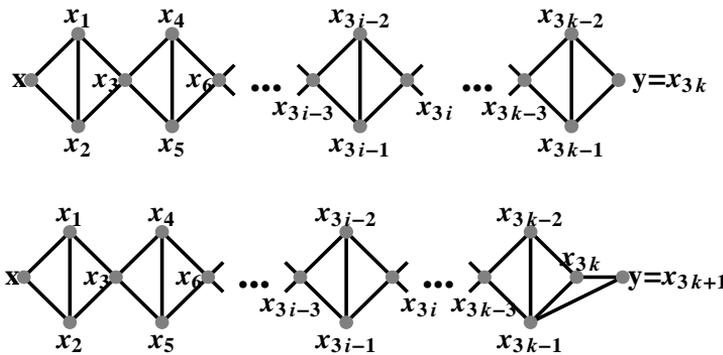


Figure 4: (a) Vertex y in CC_x , (b) Vertex y not in CC_x .

Corollary 3.7. *Let G be a 2-connected, outerplane near-triangulation with exactly two degree-2 vertices and with notation as above. If $|L(x)|, |L(y)| \leq 3$ and all other vertices have 3-lists, then there is a set of noncoloring lists if and only if*

1. $|L(x)| = |L(y)| = 1$, and the lists satisfy the conditions of Prop. 3.6 (1) or (2), or
2. $|L(x)| = 1, |L(y)| = 2$ (or vice-versa), and the lists satisfy the conditions of Prop. 3.6 (1).

Proof. If conditions (1) or (2) hold, then there is a set of noncoloring lists.

Conversely assume there is a set of noncoloring lists. We have noted that if either $L(x)$ or $L(y)$ contains three colors, then the sequential list-coloring algorithm, beginning with y or x , respectively, always extends to the whole graph. If $|L(x)| = 1$ and $|L(y)| \leq 2$, then by Prop. 3.6 the statement follows.

Suppose $|L(x)| = |L(y)| = 2$ with $L(x) = \{a, b\}$, and furthermore suppose that beginning by coloring x with color a does not lead to a list-coloring of G . By Prop. 3.6,

then $y \in CC_x$ and the lists are noncoloring. Then Lemma 3.4 and Prop. 3.6 tell us that the graph contains a chain of diamonds, as in Figure 4, and also tell us explicitly about the lists on this subgraph. Here are the noncoloring lists on the chain of diamonds subgraph, given by Lemma 3.4 and Prop. 3.6 with labels as in Figure 4. Each list for $x_i, i > 0$, is a triple of distinct colors, but otherwise the colors may repeat. As in Figure 3(b), $L(x_0) = \{a\}$, $L(x_1) = \{a, b, c\} = L(x_2)$, $L(x_3) = \{a_1, b, c\}$, $L(x_4) = \{a_1, b_1, d_1\} = L(x_5)$, $L(x_6) = \{a_2, b_1, d_1\}$, and in general for $i = 2, \dots, k - 1$ $L(x_{3i}) = \{a_i, b_{i-1}, d_{i-1}\}$, $L(x_{3i+1}) = \{a_i, b_i, d_i\} = L(x_{3i+2})$. Since these are noncoloring lists, $L(y) = \{b_{k-1}, d_{k-1}\}$.

Next we assume that initially coloring x with color b also leads to a non-list-colorable graph; the conditions of Lemma 3.4 and Prop. 3.6 hold and constrain the lists on the chain of diamonds. With x colored with b , then b does lie in both $L(x_1)$ and $L(x_2)$, and so that there is only one possible color for $L(x_3)$, we must have $a_1 = a$ so that $L(x_3) = \{a, b, c\}$. Continuing we must have $b_1 = b$ and $L(x_4) = \{a, b, d_1\} = L(x_5)$, and we must have $a_2 = a$ so that $L(x_6) = \{a, b, d_1\}$. Thus for $i = 1, \dots, k - 2$, $a_i = a_{i-1} = a$ and $b_i = b_{i-1} = b$. In the final two triangles (before and including y) $L(x_{3k-3}) = \{a_{k-1}, b_{k-2}, d_{k-2}\} = \{a, b, d_{k-2}\}$, $L(x_{3k-2}) = \{a, b, d_{k-1}\} = L(x_{3k-1})$. But now the two possible colors on x_{3k-2} and x_{3k-1} are $\{a, d_{k-1}\}$ so that $b = b_{k-1}$ can be used on y ; note that $\{a_{k-1}, b_{k-1}, d_{k-1}\} = \{a, b, d_{k-1}\}$ are the triple of colors given to $L(x_{3k-2})$ and so are distinct. \square

Theorem 3.3 follows from Cor. 3.7.

4 $\{i, j\}$ -extendability of 2-connected outerplane graphs, $i, j \leq 2$

There are some outerplane graphs that are not near-triangulations whose list-colorability follows immediately from the work of Section 3. Let G be a graph that consists of a chain of diamonds joining vertices x and x_{3k} with labels as in Figure 4. Suppose that for $i = 1, \dots, k - 1$, there is a path joining x_{3i-1} and x_{3i+2} or joining x_{3i-2} and x_{3i+1} with x_{3i} adjacent to some of the vertices of this path. In addition there may be a vertex y adjacent to x_{3k} and also joined by a path to one neighbor of x_{3k} ; x_{3k} may be adjacent to some vertices of this path, besides its endpoints. Call such a graph a *supplemented chain of diamonds*; see Figure 5 for an example when $y = x_{3k}$. The list-colorability of this graph depends on and only on the list-colorability of its chain of diamonds subgraph. Thus the relevant question is whether there is a 2-connected outerplane graph that is not a near-triangulation, contains no chain of diamonds from x to y , and whose list-coloring does not extend; the answer is no. See Figure 1(d) for an example that does not contain a chain of diamonds and Figure reffig:5 for one that does contain a chain of diamonds. It is easy to check that the former graph is $\{1, 1\}$ -extendable and that there are lists making the latter not list-colorable by continuing the lists as shown.

On the whole, except as described above, when G is not a near-triangulation, it is easier to list-color. Notice that if G consists of a single cycle with two nonadjacent, precolored vertices, then the precoloring always extends when there are 3-lists on all other vertices. Thus we assume that G has at least two interior, finite faces. As before when $|L(v)| \leq 2$ for $v = x, y$, and $|L(v)| = 3$ for all $v \in V(G) \setminus \{x, y\}$, by Prop. 3.2 we may assume that $G = G_{x,y}$ so that x and y have degree 2 and lie on different faces of G . There may be additional vertices of degree 2, but now it must be the case that G can be extended (by adding edges) to a 2-connected, outerplane near-triangulation in which x and y are the two and only two vertices of degree 2. This is possible if and only if every interior

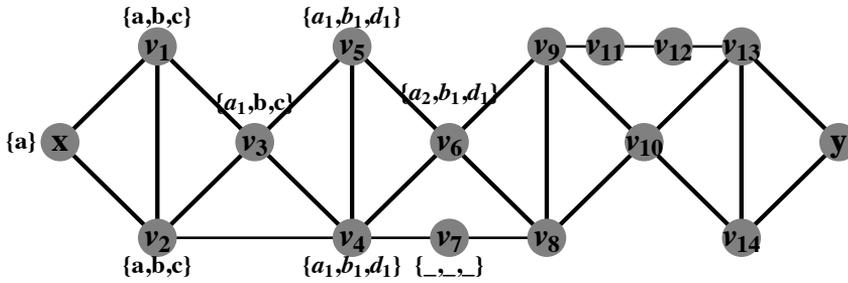


Figure 5: A labeled non-near-triangulation that contains a chain of diamonds.

edge of $G(= G_{x,y})$ separates x and y ; i.e., the removal of each interior edge and its two end-vertices leaves x and y in different components; see Figure 1(c), (d).

Now we label vertices similarly; see Figure 5 for an example. Let $x = v_0$ lie on a region R with $j + 1$ vertices; the region has exactly one interior edge e_R that lies on an additional region R' . We label the other vertices of R with v_1, \dots, v_j in any manner provided that when a vertex is labeled, it has a labeled neighbor of lower index and provided that v_j lies on e_R . Then we label that region R_j and assume that the two end-vertices of $e_R = e_{R_j}$ are v_j and v_{j_1} , $j_1 < j$. In the region R' that is incident with R_j along edge e_{R_j} , we begin the labeling from v_j , using the same procedure as for R_j , until we reach the edge $e_{R'}$, if it exists, and we conclude the labeling with largest indexed vertex, say $v_{j'}$, lying on $e_{R'}$. This second region becomes $R_{j'}$ with edge $e_{R_{j'}}$ having endpoints $v_{j'}$ and $v_{j'_1}$, $j'_1 < j'$. The only variation is that in the final region we choose to label y with v_{n-1} and then the final region R_{n-1} . As before the sequential list-coloring algorithm list-colors $G \setminus \{y\}$ by coloring vertices in order of index, always possible since every vertex of $G \setminus \{x, y\}$ has a 3-list. Note that regardless of the labeling, diagonals can be added to each interior region that is not a triangle so that in the end the graph becomes G_t , which is a near-triangulation with x and y being the vertices of degree 2.

Following the sequential list-coloring algorithm on $G \setminus \{y\}$, we define $C(i)$ and $m(i) = |C(i)|$, $i = 0, \dots, n - 2$, as before. Since G is a subgraph of G_t , it follows that $C_t(i) \subseteq C(i)$ and $m_t(i) \leq m(i)$ for $i = 0, \dots, n - 2$, where $C_t(i)$ and $m_t(i)$ are the corresponding C -sets and m -values of the near-triangulation G_t . Then by the results of Section 3, we have the following.

Lemma 4.1. *Let G be a 2-connected outerplane graph, let $x, y \in V(G)$ be nonadjacent, lying on two different interior faces, and such that every interior edge of G separates x and y . If $|L(v)| \leq 2$ for $v = x, y$, $|L(v)| = 3$ for $v \in V(G) \setminus \{x, y\}$, and with labeling as above, then*

1. *if vertex v_i has exactly one predecessor v_j , then $m(i) \geq 2$ and $m(i) = 2$ if and only if $m(j) = 1$,*
2. *if vertex v_i has two nonadjacent predecessors $v_j, v_{j'}$, then $m(i) \geq 2$ and $m(i) = 2$ if and only if $m(j) = 1$ or $m(j') = 1$,*
3. *there is no edge of G joining two vertices with m -value 1,*
4. *if R is an interior region of G , then at most one vertex of $V(R)$ has m -value 1,*

5. if $m(i) < 3$, then v_i lies on a triangle with vertices $v_{i_1}, v_{i_2}, i_1, i_2 < i, C(i_1) = C(i_2)$ with $m(i_1) = 2$, and $C(i_1) \cap L(v_i) \neq \emptyset$,
6. if there is an edge $v_j v_i, j < i$, with equal C -sets of size 2, then there is a vertex $v_k, k < i$, that forms a triangle with v_j and v_i , and $m(k) = 1$,
7. if there is an edge of G joining two vertices with m -values 1 and 2, then their C -sets are disjoint, and
8. if there is an edge $v_j v_i, j < i$, with $C(i) \neq C(j)$ and $m(i) = m(j) = 2$, then $C(i)$ and $C(j)$ are disjoint.

Proof. Many of these statements are immediate.

Statement (3) follows since $m_t(i) \leq m(i)$ and in a near-triangulation there can be no adjacent vertices with m -values 1.

For statement (4) suppose R_j is a region, $j \leq n - 1$, with its least indexed vertex being $v_{i_1}, 0 < i_1 < j$, lying also on R_i and the interior edge $v_i v_{i_1}$ between R_i and R_j for some $i < j$. If $m(i), m(i_1) \geq 2$, then R_j contains no vertex with m -value 1. By (3) without loss of generality we may assume $m(i) = 1$ and $m(i_1) \geq 2$. Then all other vertices have m -value at least 2. The same argument holds when R_j is the region with least index and contains vertex v_0 .

(5) If $m(i) = 1$ and v_i has two predecessors, then by (2) they are adjacent, forming a triangle. As we've seen in Lemma 3.4, this implies that the conditions of (5) are met. If $m(i) = 2$, then by (2), (3) v_i has two predecessors that form a triangle, implying that the conditions of (5) are met.

(6), (7), (8) The proof is the same as for Lemma 3.4(4) and Cor 3.5(1), (3). □

Proposition 4.2. *Let G be a 2-connected outerplane graph, let $x, y \in V(G)$ be nonadjacent, lying on two different faces, and such that every interior edge of G separates x and y . If $|L(v)| \leq 2$ for $v = x, y$, and $|L(v)| = 3$ for $v \in V(G) \setminus \{x, y\}$, with labeling as above, then G is not list-colorable if and only if*

1. G contains a chain of diamonds joining x and y , labeled as in Figure 4, $m(3i) = 1$ for $i = 1, \dots, k - 1$, and if $y = x_{3k}$, then $C(3k - 1) = C(3k - 2), m(3k - 1) = 2$, and $L(y) \subseteq C(3k - 1)$, or
2. G contains a chain of diamonds joining x and v_{3k} , labeled as in Figure 4(b), with y adjacent to x_{3k} , $m(3i) = 1$ for $i = 1, \dots, k$, and $|L(y)| = 1$ with $C(3k) = L(y)$.

The proof is immediate from Lemma 4.1 and the proof techniques of Prop. 3.6 and Cor. 3.7.

Theorem 4.3. *Let G be a 2-connected outerplane graph, let $x, y \in V(G)$ be nonadjacent, lying on two different faces, and such that every interior edge of G separates x and y .*

1. G is not $\{1, 1\}$ -extendable with respect to x and y if and only if G contains a chain of diamonds joining x with y or x with a neighbor of y , and the conditions on lists of Prop. 4.2 are satisfied.
2. G is not $\{1, 2\}$ -extendable with respect to x and y if and only if G contains a chain of diamonds joining x with y and the conditions on lists of Prop. 4.2(1) are satisfied.
3. G is $\{2, 2\}$ -extendable with respect to x and y .

Corollary 4.4. *Let G be a 2-connected, outerplane graph that is not a near-triangulation, let $x, y \in V(G)$ be nonadjacent, lying on two different faces, and such that every interior edge of G separates x and y . Then G is $\{i, j\}$ -extendable for all $i, j \leq 2$, unless G contains a supplemented chain of diamonds with $i = 1$ and $j \leq 2$.*

5 $\{i, j\}$ -extendability of outerplane graphs with cut-vertices

Just as Theorems 3.3 and 4.3 show the similarity of results for 2-connected, outerplane near-triangulations and non-near-triangulations, so are the related results for 1-connected outerplane graphs analogous. We present the details for the near-triangulation case.

Definition 5.1. Let x and y be nonadjacent vertices of an outerplane near-triangulation G that is connected, but contains cut-vertices. Let T_{bc} be its block-cutpoint tree with x lying in block B_i , y lying in B_j , and $p_{x,y}$ the shortest path in T_{bc} from B_i to B_j so that $p_{x,y} = B_i, v_i, \dots, B_{j-1}, v_{j-1}, B_j$ with v_i, v_{j-1} being cut-vertices on this shortest path. Then the *fundamental $x - y$ subgraph* $G_{x,y}$ consists of the triangles, their vertices and edges that comprise a shortest path of triangles in B_i from x to v_i , then for each block B_k , $i + 1 \leq k \leq j - 1$, a shortest path of triangles from v_{k-1} to v_k , and ending with a shortest path of triangles in B_j from v_{j-1} to y .

Proposition 5.2. *Let x and y be nonadjacent vertices of a connected, outerplane near-triangulation G with cut-vertices and with fundamental $x - y$ subgraph $G_{x,y}$. If $|L(v)| \leq 2$ for $v = x, y$, and $|L(v)| = 3$ for $v \in V(G) \setminus \{x, y\}$, then G can be list-colored if and only if $G_{x,y}$ can be.*

As in Prop. 3.2, the proof is an application of Thm. 1.1.

Now suppose that the outerplane near-triangulation G and its nonadjacent, degree-2 vertices x, y (with $G = G_{x,y}$) consist of $x = v_0$ in block B_1 with cut-vertex v_1 , which is also in block B_2 , block B_2 with cut-vertex v_2 , which is also in block B_3, \dots , cut-vertex v_k , which is also in block B_{k+1} , and block B_{k+1} that contains $y = v_{k+1}$. Suppose $|L(x)| = 1$ and for all $v \in V(G) \setminus \{x, y\}$, $|L(v)| = 3$. Then by Prop. 3.6 if $v_1 \in CC_x$, there are lists for $V(B_1)$ in which there are two forbidden colors on v_1 (meaning they prevent list-coloring) and so effectively in blocks B_1 and B_2 v_1 has a reduced 1-list of possible colors. If $v_1 \notin CC_x$, then there are lists for $V(B_1)$ in which there is one forbidden color on v_1 so that in blocks B_1 and B_2 v_1 has a reduced 2-list of possible colors. Thus in each block B_i , which shares the cut-vertex v_{i-1} with B_{i-1} , though v_{i-1} begins with a 3-list $L(v_{i-1})$, $L(v_{i-1})$ might be reduced to a 1- or 2-list because of non-list-colorings on B_{i-1} . If $L(v_{i-1})$ is reduced to a 2-list and $v_i \notin CC_{v_{i-1}}$, then by Prop. 3.6, block B_i is always list-colorable and $L(v_i)$ remains a 3-list. If $L(v_i)$ remains a 3-list for some $i > 0$, then the remaining graph and the entire graph is list-colorable when $|L(y)| \geq 1$ by Thm. 1.1. We deduce the following; call x, y , and the cut-vertices of G the *special vertices* of G so that each block contains two special vertices.

Theorem 5.3. *Let x and y be nonadjacent vertices of a connected outerplanar near-triangulation $G = G_{x,y}$ with cut-vertices, labeled as above. If $|L(v)| \leq 2$ for $v = x, y$ and $|L(v)| = 3$ for all $v \in V(G) \setminus \{x, y\}$, then G cannot be list-colored if and only if for at most one block B_i with special vertices v_{i-1} and v_i , $v_i \notin CC_{v_{i-1}}$. More precisely,*

1. G is not $\{1, 1\}$ -extendable with respect to x and y if and only if for exactly one block B_i with special vertices v_{i-1} and v_i , $v_i \notin CC_{v_{i-1}}$.

2. G is not $\{1, 2\}$ -extendable with respect to x and y if and only if for every block B_i with special vertices v_{i-1} and v_i , $v_i \in CC_{v_{i-1}}$.
3. G is always $\{2, 2\}$ -extendable with respect to x and y .

Proof. Suppose G is not list-colorable, $|L(x)| = 1$, and there is a block B_i for which $v_i \notin CC_{v_{i-1}}$; let i be the first index for which the latter holds. Then the reduced list-size for v_{i-1} is 1 since two colors were forbidden on v_{i-1} in B_{i-1} . Then by Prop. 3.6 since $v_i \notin CC_{v_{i-1}}$, there is at most one forbidden color for v_i , and $L(v_i)$ is reduced to two colors for B_i and B_{i+1} . Then in B_{i+1} and in every subsequent block we must have $v_{i+1} \in CC_{v_i}$ and $|L(y)| = 1$ to prevent list-coloring. If there is no such block B_i , then condition (2) holds.

If G is not list-colorable and $|L(x)| = |L(y)| = 2$, then at v_1 there is at most one forbidden color so that $L(v_1)$ has a reduced list of size 2. Then it follows that for each cut vertex v_i the reduced list is of size at least 2, and the final block has both special vertices having lists of size at least 2. By Thm. 3.3 G would be list-colorable. \square

We note that Lemma 5 of [8] gives an instance of Thm. 5.3(3) on the limited class of outerplanar graphs called "generalized (broken) wheel strings," though there a slightly stronger conclusion is reached.

A theorem analogous to Thm. 5.3 holds for outerplane graphs that are not near-triangulations, just as the conclusions of Thm. 3.3 hold for the conditions of Thm. 4.3, with the presence of a chain of diamonds playing the role of CC_x .

6 Some general results and questions

These results on outerplane graphs lend evidence for an affirmative answer to the following question about arbitrary planar graphs:

Conjecture 6.1. *Is every planar graph $\{2, 2\}$ -extendable?*

Let $G[V']$ denote the induced subgraph on $V' \subseteq V(G)$.

Corollary 6.2. *Let G be a plane near-triangulation with outer cycle C . Let x and y be nonadjacent vertices of C , and suppose $|L(v)| \leq 2$ for $v = x, y$, $|L(v)| \geq 3$ for $v \in V(C) \setminus \{x, y\}$, and $|L(v)| \geq 5$ for $v \in V(G) \setminus V(C)$.*

1. *When $G[C]$ is a near-triangulation, G is L -list-colorable if and only if $G[C]$ is.*
2. *When all interior faces of $G[C]$ have at most six-edges, G is L -list-colorable if and only if $G[C]$ is except possibly when the interior of some face contains a vertex adjacent to at least five vertices on the face boundary, or contains an edge or a triangle whose vertices are adjacent to six vertices on the face boundary.*
3. *G is L -list-colorable when $|L(v)| \geq 4$ for $v \in V(C) \setminus \{x, y\}$.*
4. *When $|L(v)| = 2$ for $v = x$ and y , G is L -list-colorable when in one subpath of C joining x and y , there is at most one vertex with a 3-list and all others on this subpath have 4-lists.*
5. *When $|L(v)| = 2$ for $v = x$ and y , G is L -list-colorable when $x = v_1, v_2, y = v_3$ are three consecutive vertices of C .*

Part (1) follows from Thm. 1.1. Part (2) follows from [5, 8]. Parts (3) and (4) follow from the results of [3], and Part (5) is a corollary of Part (4). More explicitly for Part (3), suppose $|L(v)| \geq 4$ for $v \in V(C) \setminus \{x, y\}$. Color x with one of $L(x)$, say color a . Delete x and delete color a , when present, from $L(v)$ for every neighbor v of x . Since the resulting graph satisfies the hypotheses of Thm. 1.1, it can be list-colored and that coloring gives a list-coloring of G with x colored a ; see [3].

Some additional results on the general case of Conj. 19 have been obtained by Axenovich and Lastrina [4].

References

- [1] M. Aigner and G. M. Ziegler, *Proofs from THE BOOK*, Springer, Berlin, 1991.
- [2] M. O. Albertson, You can't paint yourself into a corner, *J. Combinatorial Theory, Ser. B*, **73** (1998), 189–194.
- [3] M. Axenovich, J. P. Hutchinson and M. A. Lastrina, List precoloring extension in planar graphs, *Discrete Math.* **311** (2011), 1046–1056.
- [4] M. Axenovich and M. A. Lastrina, personal communication, 2010.
- [5] T. Böhme, B. Mohar and M. Stiebitz, Dirac's map-color theorem for choosability, *J. Graph Theory* **32** (1999), 327–339.
- [6] B. Mohar and C. Thomassen, *Graphs on Surfaces*, Chapter 7, Johns Hopkins University Press, Baltimore, 2001.
- [7] C. Thomassen, Every planar graph is 5-choosable, *J. Combinatorial Theory, Ser. B* **62** (1994), 180–181.
- [8] C. Thomassen, Exponentially many 5-list-colorings of planar graphs, *J. Combinatorial Theory, Ser. B* **97** (2007), 571–583.
- [9] Zs. Tuza and M. Voigt, A note on planar 5-list colouring: non-extendability at distance 4, *Discrete Math.* **251** (2002), 169–172.
- [10] D. B. West, *Introduction to Graph Theory, 2nd ed.*, Prentice-Hall, Upper Saddle River, NJ, 2001.